1. An airplane accelerates at 10 mph per second, and decelerates at 15 mph/sec. Given that its takeoff speed is 180 mph, and the pilots want enough runway length to safely decelerate to a stop from any speed below takeoff speed, what's the shortest length that the runway can be allowed to be? Assume the pilots always use maximum acceleration when accelerating. Please give your answer in miles.

   Answer: $\frac{3}{4} = 0.75$ miles.

   Solution: In the worst case, where the plane is just shy of takeoff speed before decelerating, the average speed of the plane will be 90 mph. It takes 18 seconds to accelerate all the way, and 12 seconds to decelerate. Therefore, it spends 30 seconds at an average of 90 miles per hour, or $\frac{90}{3600}$ miles per second, which means that it travels $30 \times \frac{90}{3600} = \frac{3}{4}$ miles in this worst case, so $\boxed{\frac{3}{4}}$ is our answer.

2. If there is only 1 distinct complex solution to the equation

$$8x^3 + 12x^2 + kx + 1 = 0$$

   what is k?

   Answer: 6

   Solution: Since the cubic has only one distinct complex root, it must be of the form $8(x-r)^3$. Comparing the x^2 coefficients gives $r = -\frac{1}{2}$, and expanding $8(x+\frac{1}{2})^3$ gives $k = \boxed{6}$.

3. If f is a polynomial, and $f(-2) = 3$, $f(-1) = -3 = f(1)$, $f(2) = 6$, and $f(3) = 5$, then what is the minimum possible degree of f?

   Answer: 4

   Solution: Using the division algorithm, we find that

$$f(x) = (x^2 - 1)g(x) - 3$$

   for some polynomial g. It remains to show that g must have degree at least 2. To do this, we see that, plugging in -2, 2, and 3, we have

$$g(2) = 3, g(-2) = 2, g(3) = 1$$

   Obviously, g is not constant, and we have

$$\frac{g(2) - g(-2)}{2 - (-2)} = \frac{1}{4} \neq -2 = \frac{g(2) - g(3)}{2 - 3}$$

   so g is not linear either. Hence g has degree at least 2, so f has degree at least 4. A quadratic through the three listed points exists, so degree 4 is attainable and the answer is $\boxed{4}$.

4. Find

$$\sum_{i=1}^{2016} i(i+1)(i+2) \pmod{2018}$$

   Answer: 0

   Solution: Let $u_i = i(i+1)(i+2)$. Then $iu_{i+1} = (i+3)u_i$, so $iu_{i+1} - (i-1)u_i = 4u_i$. Summing both sides as i ranges from 1 to n, we have

$$nu_{n+1} = 4\sum_{i=1}^{n} i(i+1)(i+2).$$

Calling the sum S_n and putting $n = 2016$, we have

$$S_{2016} = \frac{2016(2017)(2018)(2019)}{4} \equiv \boxed{0} \pmod{2018}.$$

5. Find the product of all complex values of d such that $x^3 + 2x^2 + 3x + 4 = 0$ and $x^2 + dx + 3 = 0$ have a common root.

Answer: 1

Solution:

Solution 1: If you haven't encountered the notion of eliminant before, that's okay. Here's how you'd proceed: Let $x + k$ be the factor corresponding to a common root. Then

$$x^3 + 2x^2 + 3x + 4 = (x + k)(x^2 + lx + m)$$

$$x^2 + dx + 3 = (x + k)(x + n)$$

for some l, m, n. Now

$$(x^3 + 2x^2 + 3x + 4)(x + n) = (x^2 + dx + 3)(x^2 + lx + m)$$

Equate like powers of x to get a system of equations to solve for l, m, n; find the determinant of the coefficient matrix of this system, and set it equal to zero. This will give you a polynomial equation in d: the negative of the constant term divided by the coefficient of the cubic term will yield the required quantity. The determinant turns out to be

$$\begin{vmatrix} 1 & 0 & 1 & d-2 \\ d & 1 & 2 & 0 \\ 3 & d & 3 & -4 \\ 0 & 3 & 4 & 0 \end{vmatrix}$$

The cubic term appears only in the term containing the product $(d)(d)(d-2)$, hence its coefficient is -4. Cofactor-expanding via the last column, we look for terms not containing d to speed up the calculation: we get

$$-(-4)\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} - (-2)\left(-3\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}\right) = -8 + 12 = 4.$$

The answer is thus $-4/(-4) = \boxed{1}$.

*Note that there is a way to do this and get a determinant of lower order (cf. the method of Bézout/Cauchy), but I give what I feel is the intuitively fastest way.

Solution 2: Alternatively, one can use long division to compute that

$$x^3 + 2x^2 + 3x + 4 = (x + (2 - d))(x^2 + dx + 3) + d(d - 2)x + 3d - 2.$$

Thus a common root must also make the remainder $d(d-2)x + 3d - 2$ equal to zero. The cases $d = 0$ and $d = 2$ give nonzero constant remainders, so otherwise the common root must be

$$\frac{2 - 3d}{d(d - 2)}.$$

Plugging that in the quadratic and setting it to 0, we have

$$(2 - 3d)^2 + d^2(2 - 3d)(d - 2) + 3d^2(d - 2)^2 = 0$$

Notice that the d^4 terms cancel out, and we see that the coefficient of d^3 is -4 and the constant term is 4, so by Vieta's formulas the product is $\frac{-4}{-4} = \boxed{1}$ as desired.

6. Let $x, y, z \in \mathbb{R}$ and
$$7x^2 + 7y^2 + 7z^2 + 9xyz = 12$$
The minimum value of $x^2+y^2+z^2$ can be expressed as $\frac{a}{b}$ where $a, b \in \mathbb{Z}$, $b > 0$, and $\gcd(a, b) = 1$. What is $a + b$?

Answer: 7

Solution: Let $S = x^2 + y^2 + z^2$. If $xyz \leq 0$, then from
$$7S + 9xyz = 12$$
we have $S \geq \frac{12}{7} > \frac{4}{3}$. If $xyz > 0$, AM-GM applied to x^2, y^2, z^2 gives
$$xyz \leq \left(\frac{S}{3}\right)^{3/2}.$$
Hence
$$12 = 7S + 9xyz \leq 7S + 9\left(\frac{S}{3}\right)^{3/2}.$$
If $S < \frac{4}{3}$, the right-hand side is strictly less than its value at $S = \frac{4}{3}$, which is 12. Thus $S \geq \frac{4}{3}$. To see that equality is attainable, set $x = y = z$; dividing the original equation by 3 gives
$$3x^3 + 7x^2 - 4 = 0.$$
The left-hand side factors as
$$(x + 1)(x + 2)(3x - 2) = 0.$$
Taking $x = y = z = \frac{2}{3}$ gives $S = \frac{4}{3}$, so the minimum value is $\frac{4}{3}$ and the answer is $4 + 3 = \boxed{7}$.

7. Let
$$h_n := \sum_{k=0}^{n} \binom{n}{k} \frac{2^{k+1}}{(k+1)}$$
Find
$$\sum_{n=0}^{\infty} \frac{h_n}{n!}.$$

Answer: $e^3 - e$

Solution:
$$\sum_{k=0}^{n} \binom{n}{k} \frac{2^{k+1}}{(k+1)} = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k+1} 2^{k+1} = \frac{3^{n+1} - 1}{n+1}$$
$$\sum_{n=0}^{\infty} \frac{h_n}{n!} = \sum_{n=0}^{\infty} \frac{3^{n+1}}{(n+1)!} - \sum_{n=0}^{\infty} \frac{1}{(n+1)!} = (e^3 - 1) - (e - 1) = \boxed{e^3 - e}$$

8. Compute
$$\sum_{k=1}^{1009} (-1)^{k+1} \binom{2018 - k}{k - 1} 2^{2019-2k}$$

Answer: 2018

Solution:

$$\begin{aligned}
\sum_{k=1}^{[n/2]+1} (-1)^{k+1}\binom{n-k}{k-1}2^{n-2k+1} &= \text{coefficient of } x^n \text{ in } \sum_{k=1} x^{2k-1}(2x-1)^{n-k} && (1)\\
&= \text{coefficient of } x^n \text{ in } \frac{x(2x-1)^{n-1}}{1-\frac{x^2}{2x-1}} && (2)\\
&= \text{coefficient of } x^n \text{ in } \frac{-x(2x-1)^n}{(x-1)^2} && (3)\\
&= \text{coefficient of } x^n \text{ in } \frac{-x(x+(x-1))^n}{(x-1)^2} && (4)\\
&= \text{coefficient of } x^n \text{ in } n\frac{x^n}{1-x} && (5)\\
&= n && (6)
\end{aligned}$$

Hence, putting $n = 2018$, we get

$$\sum_{k=1}^{1009}(-1)^{k+1}\binom{2018-k}{k-1}2^{2019-2k} = \boxed{2018}$$

9. Suppose

$$\frac{1}{5}\frac{(x+1)(x-3)}{(x+2)(x-4)} + \frac{1}{9}\frac{(x+3)(x-5)}{(x+4)(x-6)} - \frac{2}{13}\frac{(x+5)(x-7)}{(x+6)(x-8)} = \frac{92}{585}$$

Also, suppose $x > 0$. Then x can be written as $a + \sqrt{b}$ where a, b are integers. Find $a + b$.

Answer: 20

Solution: Put $x = y + 1$ and substitute. Then

$$\frac{1}{5}\left(\frac{y^2-4}{y^2-9}\right) + \frac{1}{9}\left(\frac{y^2-16}{y^2-25}\right) - \frac{2}{13}\left(\frac{y^2-36}{y^2-49}\right) = \frac{92}{585}$$

Also note that

$$\frac{1}{5} + \frac{1}{9} - \frac{2}{13} = \frac{92}{585}$$

Subtracting the above equations, we get

$$\frac{1}{y^2-9} + \frac{1}{y^2-25} - \frac{2}{y^2-49} = 0$$

i.e.,

$$\left(\frac{1}{y^2-9} - \frac{1}{y^2-49}\right) + \left(\frac{1}{y^2-25} - \frac{1}{y^2-49}\right) = 0$$

Thus

$$\frac{5}{y^2-9} = \frac{-3}{y^2-25}.$$

Therefore, $y^2 = 19$, so $x = 1 + \sqrt{19}$ (since $x > 0$). Thus $a = 1$, $b = 19$, so $a + b = \boxed{20}$.

10. Let a, b, c be the roots of the equation $x^3 - 2018x + 2018 = 0$. Let q be the smallest positive integer for which there exists an integer p, $0 < p \leq q$, such that

$$\frac{a^{p+q} + b^{p+q} + c^{p+q}}{p+q} = \left(\frac{a^p + b^p + c^p}{p}\right)\left(\frac{a^q + b^q + c^q}{q}\right)$$

Find $p^2 + q^2$.

Answer: 13

Solution: Observe that $a+b+c = 0$. Now, we get $(1+ax)(1+bx)(1+cx) = 1 - 2018x^2 - 2018x^3$. Take the (formal) natural logarithm of both sides; expanding as a series and equating coefficients of like powers of x, we get

$$-\frac{a^2 + b^2 + c^2}{2} = -2018, \qquad \frac{a^3 + b^3 + c^3}{3} = -2018, \qquad -\frac{a^4 + b^4 + c^4}{4} = -\frac{(2018^2)}{2},$$

and

$$\frac{a^5 + b^5 + c^5}{5} = -2018^2.$$

Checking $q = 1$ and $q = 2$ from these values shows that no admissible p works, while $p = 2$, $q = 3$ does work. Thus the required p, q are 2, 3, and $p^2 + q^2 = \boxed{13}$.