1. Real numbers x, y, and z satisfy
$$5x + 6y + 7z = 0,$$
$$6x + 7y + 8z = 1.$$
Compute $65x + 66y + 67z$.

Answer: 60

Solution: Subtracting the first equation from the second equation gives
$$x + y + z = 1.$$
We can rewrite the desired expression as
$$\begin{aligned} 65x + 66y + 67z &= 60(x + y + z) + (5x + 6y + 7z) \\ &= 60(1) + 0 \\ &= \boxed{60}. \end{aligned}$$

2. A hat initially contains five slips labeled with the numbers 1, 2, 3, 4, and 5 (one slip per number). Every minute, Jessica adds a new slip labeled with the number 20 to the hat. How many slips are in the hat when the average of the numbers in the hat is 5 times the initial average?

Answer: 17

Solution: The initial average is
$$\frac{1 + 2 + 3 + 4 + 5}{5} = \frac{15}{5} = 3.$$
Let x be the number of minutes until the average of the numbers is 5 times the initial average. We can write the following equation and solve for x:
$$\begin{aligned} 5 \cdot 3 &= \frac{15 + 20x}{5 + x} \\ 15 &= \frac{15 + 20x}{5 + x} \\ 15 \cdot (5 + x) &= 15 + 20x \\ 75 + 15x &= 15 + 20x \\ 60 &= 5x \\ 12 &= x. \end{aligned}$$
Therefore, it takes 12 minutes until the average is five times the initial average. The hat started with 5 slips and 12 more are added, so at the end there are $\boxed{17}$ slips in the hat.

3. An arithmetic sequence is a sequence of numbers where the difference between any two consecutive terms is the same. Harsh writes an arithmetic sequence where the ratio of the second term to the first term is 4, and the sum of the third and fourth terms is 34. Compute the second term of Harsh's sequence.

Answer: 8

Solution: Let the first term be a, and let the common difference be d. Then the second, third, and fourth terms are $a + d$, $a + 2d$, and $a + 3d$, respectively. We are given
$$\frac{a + d}{a} = 4 \quad \text{and} \quad (a + 2d) + (a + 3d) = 34.$$

We can solve for d in terms of a from the first equation:

$$\begin{aligned}\frac{a+d}{a} &= 4\\ a+d &= 4a\\ d &= 3a.\end{aligned}$$

Plugging this into the second equation gives

$$\begin{aligned}(a+2d)+(a+3d) &= 34\\ 2a+5d &= 34\\ 2a+5\cdot(3a) &= 34\\ 17a &= 34\\ a &= 2.\end{aligned}$$

Using $d=3a$ and $a=2$ gives $d=6$. Thus, the second term of the sequence is $a+d=\boxed{8}$.

4. Let $f(x)=(2x^2+7x+3)(x^2-2x-15)$. The graph of $f(x)$ in the coordinate plane is translated 4 units to the right and then rotated about the origin by 180 degrees. The resulting graph is the graph of the function $g(x)$. Compute the sum of the distinct roots of $g(x)$.

   Answer: $\mathbf{-\frac{27}{2}}$ **or** $\mathbf{-13.5}$

   Solution: Observe that $f(x)$ can be factored as $f(x)=(2x+1)(x+3)(x+3)(x-5)$. Therefore, the distinct roots of $f(x)$ are $-\frac{1}{2}, -3$, and 5. When f is translated 4 units to the right, each of the intersections with the x-axis is translated to the right by 4 units. When the resulting graph is rotated about the origin by 180 degrees, each intersection with the x-axis is effectively reflected about the y-axis. Neither of the two transformations above introduces any additional intersection points with the x-axis. Thus, each root r of f corresponds to the root $-(r+4)$ of g; this accounts for all roots of g.

   The distinct roots of $g(x)$ are $\{-9, -\frac{7}{2}, -1\}$, which sum to $\boxed{-\frac{27}{2}}$.

5. Complex numbers a, b, and c satisfy the equations

$$\begin{aligned}a^3+b^3 &= 2-3ab(a+b),\\ b^3+c^3 &= 16-3bc(b+c),\\ c^3+a^3 &= 54-3ca(c+a).\end{aligned}$$

   Compute the sum of all distinct possible real values of $a+b+c$.

   Answer: $\mathbf{\frac{3\sqrt[3]{2}}{2}}$

   Solution: We can rearrange the given equations and factor using the Binomial Theorem as follows:

$$\begin{aligned}a^3+b^3+3ab(a+b) &= a^3+3a^2b+3ab^2+b^3=(a+b)^3=2,\\ b^3+c^3+3bc(b+c) &= b^3+3b^2c+3bc^2+c^3=(b+c)^3=16,\\ c^3+a^3+3ca(c+a) &= c^3+3c^2a+3ca^2+a^3=(c+a)^3=54.\end{aligned}$$

The solutions are multiples of third roots of unity. Let $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$; then each of the given equations has three possible solutions:

$$\begin{aligned} a + b &= \sqrt[3]{2}, \sqrt[3]{2}\omega, \sqrt[3]{2}\omega^2, \\ b + c &= 2\sqrt[3]{2}, 2\sqrt[3]{2}\omega, 2\sqrt[3]{2}\omega^2, \\ c + a &= 3\sqrt[3]{2}, 3\sqrt[3]{2}\omega, 3\sqrt[3]{2}\omega^2. \end{aligned}$$

Note $a + b + c = \frac{(a+b)+(b+c)+(c+a)}{2}$, and we wish to pick values for $a+b$, $b+c$, and $c+a$ such that their sum is a real number. We have a few cases based on how many of $a+b, b+c,$ and $a+c$ are real:

- If all three are real, then $a+b = \sqrt[3]{2}$, $b+c = 2\sqrt[3]{2}$, and $c+a = 3\sqrt[3]{2}$, so $a+b+c = \frac{\sqrt[3]{2}+2\sqrt[3]{2}+3\sqrt[3]{2}}{2} = 3\sqrt[3]{2}$.
- If two of them are real, then the third has a nonzero imaginary part, and the sum will have the same nonzero imaginary part.
- If one of them is real, then the other two have nonzero imaginary parts. However, none of the imaginary parts are negations of each other, so the sum will have a nonzero imaginary part.
- If none of them are real, then two of the imaginary parts should sum to the negation of the third. The only ways are $(a+b, b+c, c+a) = (\sqrt[3]{2}\omega, 2\sqrt[3]{2}\omega, 3\sqrt[3]{2}\omega^2)$ and $(a+b, b+c, c+a) = (\sqrt[3]{2}\omega^2, 2\sqrt[3]{2}\omega^2, 3\sqrt[3]{2}\omega)$. Both lead to the same sum $3\sqrt[3]{2}(\omega+\omega^2) = -3\sqrt[3]{2}$, so $a+b+c = -\frac{3\sqrt[3]{2}}{2}$ in this case.

The sum of all possible real values of $a+b+c$ is $3\sqrt[3]{2} + \left(-\frac{3\sqrt[3]{2}}{2}\right) = \boxed{\frac{3\sqrt[3]{2}}{2}}$.

6. Compute the sum of all distinct real numbers x satisfying $|x| \le \frac{1}{2}$ and

$$\sin\left(\log_{10}\left(\frac{1}{x} + \frac{1}{x^2}\right)\right) = 0.$$

Note that the argument of the sine function is assumed to be in radians.

Answer: $\mathbf{\frac{1}{10^{\pi}-1}}$

Solution: Since $\sin\theta = 0$ exactly when $\theta = k\pi$ for an integer k, we have

$$\log_{10}\left(\tfrac{1}{x} + \tfrac{1}{x^2}\right) = k\pi \quad \text{for some integer } k.$$

Exponentiating gives

$$\frac{1}{x} + \frac{1}{x^2} = 10^{k\pi}.$$

The left side is undefined at $x = 0$. On $[-1/2, 0)$ it is increasing from 2 to ∞, while on $(0, 1/2]$ it is decreasing from ∞ to 6. For integers $k \ge 1$ there are two different solutions, and for integers $k \le 0$ there are no solutions. Rewrite the equation in standard quadratic form:

$$\frac{1}{x} + \frac{1}{x^2} = 10^{k\pi} \quad \Longrightarrow \quad 10^{k\pi}x^2 - x - 1 = 0.$$

Since the discriminant $\Delta = 1 + 4 \cdot 10^{k\pi}$ of this quadratic is always positive, there are two real roots for each k. By Vieta's formulas, the sum of those two roots is

$$\frac{-(-1)}{10^{k\pi}} = 10^{-k\pi}.$$

Another way to see this is by observing that the two roots are given by $\frac{1\pm\sqrt{\Delta}}{2\cdot 10^{k\pi}}$, so the sum of the two roots is $\frac{2}{2\cdot 10^{k\pi}} = \frac{1}{10^{k\pi}}$.

Finally, summing over all integers $k \geq 1$ gives the infinite geometric series

$$\sum_{k=1}^{\infty} 10^{-k\pi} = \frac{10^{-\pi}}{1-10^{-\pi}} = \boxed{\frac{1}{10^{\pi}-1}}.$$

7. Let $\omega = e^{\frac{2i\pi}{5}}$. Compute

$$\prod_{i=1}^{5} \prod_{j=i+1}^{5} (\omega^i - \omega^j)^2.$$

Answer: 3125

Solution 1: Recall that $\omega^1, \omega^2, \omega^3, \omega^4, \omega^5$ are the roots of $z^5 - 1$, so we can factor $z^5 - 1 = (z-1)(z^4+z^3+z^2+z+1) = \prod_{i=1}^{5}(z-\omega^i)$. Then the numbers $\omega^1, \ldots, \omega^4$ are the roots of

$$z^4 + z^3 + z^2 + z + 1 = \sum_{n=0}^{4} z^n$$

so the numbers $(1-\omega), \ldots, (1-\omega^4)$ are the roots of

$$\sum_{n=0}^{4} (1-z)^n.$$

The constant term of this polynomial is 5 and the coefficient of the z^4 term is 1, so

$$\prod_{i=1}^{4}(1-\omega^i) = 5$$

by Vieta's. For each $j \in \{1, 2, 3, 4, 5\}$, we have

$$\prod_{1\leq i\leq 5, i\neq j} (\omega^j - \omega^i) = \prod_{i=1}^{4}(\omega^j - \omega^{i+j}) = \omega^{4j} \prod_{i=1}^{4}(1-\omega^i) = 5\omega^{4j} = \frac{5}{\omega^j}.$$

Taking the product of the far LHS and far RHS over all $j \in \{1, 2, 3, 4, 5\}$ gives

$$\prod_{1\leq i\neq j\leq 5} (\omega^j - \omega^i) = \frac{5^5}{\prod_{j=1}^{5} \omega^j} = 5^5.$$

Finally,

$$\prod_{i=1}^{5} \prod_{j=i+1}^{5} (\omega^i - \omega^j)^2 = \prod_{1\leq i<j\leq 5} (\omega^i - \omega^j)^2 = (-1)^{10} \prod_{1\leq i\neq j\leq 5} (\omega^i - \omega^j) = 5^5 = \boxed{3125}.$$

Solution 2: Let $\omega_k = e^{\frac{2\pi i k}{5}}$. Consider the polynomial $x^5 - 1$. It has distinct roots $\omega_1, \omega_2, \dots, \omega_5$, so we can factor $x^5 - 1$ as

$$x^5 - 1 = (x - \omega_1)(x - \omega_2)(x - \omega_3)(x - \omega_4)(x - \omega_5).$$

Also, $1 = \omega_1^5 = \cdots = \omega_5^5$, so we can also factor $x^5 - 1$ as, for example,

$$x^5 - 1 = x^5 - \omega_1^5 = (x - \omega_1)(x^4 + x^3\omega_1 + x^2\omega_1^2 + x\omega_1^3 + \omega_1^4).$$

Comparing the two factorizations of $x^5 - 1$ gives

$$(x - \omega_2)(x - \omega_3)(x - \omega_4)(x - \omega_5) = x^4 + x^3\omega_1 + x^2\omega_1^2 + x\omega_1^3 + \omega_1^4.$$

Plugging in $x = \omega_1$ gives

$$(\omega_1 - \omega_2)(\omega_1 - \omega_3)(\omega_1 - \omega_4)(\omega_1 - \omega_5) = \omega_1^4 + \omega_1^3\omega_1 + \omega_1^2\omega_1^2 + \omega_1\omega_1^3 + \omega_1^4 = 5\omega_1^4.$$

Similarly, we can get:

$$\begin{aligned}
(\omega_2 - \omega_1)(\omega_2 - \omega_3)(\omega_2 - \omega_4)(\omega_2 - \omega_5) &= 5\omega_2^4, \\
(\omega_3 - \omega_1)(\omega_3 - \omega_2)(\omega_3 - \omega_4)(\omega_3 - \omega_5) &= 5\omega_3^4, \\
(\omega_4 - \omega_1)(\omega_4 - \omega_2)(\omega_4 - \omega_3)(\omega_4 - \omega_5) &= 5\omega_4^4, \\
(\omega_5 - \omega_1)(\omega_5 - \omega_2)(\omega_5 - \omega_3)(\omega_5 - \omega_4) &= 5\omega_5^4.
\end{aligned}$$

Multiplying both sides of the five equations above, we get

$$\prod_{i \neq j} (\omega_i - \omega_j) = 5^5 \omega_1^4 \omega_2^4 \omega_3^4 \omega_4^4 \omega_5^4$$

Recall that $\omega_k = e^{\frac{2\pi i k}{5}} = \omega_1^k$, so

$$\omega_1^4 \omega_2^4 \omega_3^4 \omega_4^4 \omega_5^4 = \omega_1^{4+8+12+16+20} = \omega_1^{60} = (\omega_1^5)^{12} = 1^{12} = 1.$$

Substituting this yields

$$\prod_{i \neq j} (\omega_i - \omega_j) = 5^5.$$

As in Solution 1, this is different from our desired expression by some factors of (-1); we will show more explicitly here how to convert this expression to the expression given in the problem statement.

The product in the problem statement is equivalent to $\prod_{i<j} (\omega_i - \omega_j)^2$, whereas in our expression $\prod_{i \neq j} (\omega_i - \omega_j)$, for every pair of $i < j$, we have two terms

$$(\omega_i - \omega_j)(\omega_j - \omega_i) = (-1)(\omega_i - \omega_j)^2.$$

There are 10 pairs of $i < j$ for $1 \leq i, j \leq 5$, so we are off by a factor $(-1)^{10} = 1$. The answer is still

$$\prod_{i<j} (\omega_i - \omega_j)^2 = (-1)^{10} \prod_{i \neq j} (\omega_i - \omega_j) = 5^5 = \boxed{3125}.$$

8. Let $a = \log_6(30)$ and $b = \log_{15}(24)$. Then

$$\frac{2ab + 2a - 1}{ab + b + 1} = \log_m(n)$$

for positive integers m and n such that m is as small as possible. Compute $m + n$.

Answer: 72

Solution 1: Let $6^a = 30$ and $15^b = 24$. Note that $5400 = 30^2 \cdot 6 = 15^2 \cdot 24$, so that $6^{2a+1} = 15^{b+2}$. Observe that

$$\begin{aligned}
6^{2a+1} &= 15^{b+2} \\
30^{2+\frac{1}{a}} &= 15^{b+2} \\
2^{2+\frac{1}{a}} &= 15^{b-\frac{1}{a}} \\
2^{2a+1} &= 15^{ab-1} \\
12^{2a+1} &= 15^{ab+b+1}.
\end{aligned}$$

In addition,

$$\begin{aligned}
6^{2a+1} &= 15^{b+2} \\
6^{2a+1} &= 24^{1+\frac{2}{b}} \\
6^{2a-\frac{2}{b}} &= 4^{1+\frac{2}{b}} \\
6^{2ab-2} &= 4^{b+2} \\
6^{2ab+2a-1} &= 60^{b+2}.
\end{aligned}$$

From the first line and the last line of each sequence of observations above,

$$\log_{15}(6) = \frac{b+2}{2a+1}, \;\; \log_{15}(12) = \frac{ab+b+1}{2a+1}, \;\; \log_6(60) = \frac{2ab+2a-1}{b+2}.$$

We see

$$\begin{aligned}
\frac{2ab+2a-1}{ab+b+1} &= \frac{2ab+2a-1}{b+2} \cdot \frac{b+2}{2a+1} \cdot \frac{2a+1}{ab+b+1} \\
&= \log_6(60)\log_{15}(6)\log_{12}(15) \\
&= \log_{12}(60).
\end{aligned}$$

There is no way to express $\log_{12}(60)$ as $\log_m(n)$ for smaller integers m and n, hence $m+n = \boxed{72}$.

Solution 2: We eventually want to substitute in a and b, but the current expression is not very workable since adding and multiplying arbitrary logarithms together is hard. However, adding and multiplying constants and taking reciprocals are easier. First, we will manipulate the expression to separate a and b as much as possible.

To do so, we'd like to factor parts of the expression, preferably the denominator. To that end, we can rewrite:

$$\begin{aligned}
\frac{2ab+2a-1}{ab+b+1} &= \frac{ab+b+1}{ab+b+1} + \frac{ab+2a-b-2}{ab+b+1} \\
&= 1 + \left(\frac{ab+b+1}{ab+2a-b-2}\right)^{-1} \\
&= 1 + \left(\frac{ab+b+1}{(a-1)(b+2)}\right)^{-1}.
\end{aligned}$$

So it suffices to calculate $\dfrac{ab+b+1}{(a-1)(b+2)}$. We can eliminate b from the numerator like so:

$$\begin{aligned}\frac{ab+b+1}{(a-1)(b+2)} &= \frac{ab+b+2a+2}{(a-1)(b+2)} - \frac{2a+1}{(a-1)(b+2)}\\ &= \frac{(a+1)(b+2)}{(a-1)(b+2)} - \frac{2a+1}{(a-1)(b+2)}\\ &= \frac{a+1}{a-1} - \frac{2a+1}{(a-1)(b+2)}.\end{aligned}$$

Now the fractions in the expression are simple enough that we can substitute $a = \log_6(30)$ and $b = \log_{15}(24)$. We have

$$\frac{a+1}{a-1} = \frac{\log_6(30)+\log_6(6)}{\log_6(30)-\log_6(6)} = \frac{\log_6(180)}{\log_6(5)} = \log_5(180)$$

and

$$\begin{aligned}\frac{2a+1}{(a-1)(b+2)} &= \frac{2\log_6(30)+\log_6(6)}{(\log_6(30)-\log_6(6))(\log_{15}(24)+\log_{15}(15^2))}\\ &= \frac{\log_6(5400)}{\log_6(5)\cdot\log_{15}(5400)}\\ &= \frac{\log_{5400}(15)}{\log_6(5)\cdot\log_{5400}(6)}\\ &= \frac{\log_6(15)}{\log_6(5)}\\ &= \log_5(15).\end{aligned}$$

Therefore,

$$\frac{ab+b+1}{(a-1)(b+2)} = \log_5(180) - \log_5(15) = \log_5(12).$$

We conclude that

$$\log_m(n) = 1 + (\log_5(12))^{-1} = \log_{12}(12) + \log_{12}(5) = \log_{12}(60),$$

so $m+n = 12+60 = \boxed{72}$.

9. Distinct complex numbers $a, b,$ and c each have integer real part and satisfy the equations

$$ab+bc+ca = 300,$$

$$(a-b)^5 + (b-c)^5 + (c-a)^5 = 0.$$

Compute the sum of the three least possible positive real values of abc.

Answer: 2000

Solution: Note that $(a-b)^5 + (b-c)^5 + (c-a)^5 = 0$ whenever $a=b$, $b=c$, or $c=a$, so $(a-b)(b-c)(c-a)$ divides $(a-b)^5+(b-c)^5+(c-a)^5$. Thus, we expect a factorization

$$(a-b)^5+(b-c)^5+(c-a)^5 = (a-b)(b-c)(c-a)P(a,b,c)$$

for some symmetric 3-variable polynomial P. Let $x = a - b$ and $y = b - c$. Then we can factor as follows:

$$\begin{aligned}(a-b)^5 + (b-c)^5 + (c-a)^5 &= x^5 + y^5 + (-x-y)^5 \\ &= -5x^4y - 10x^3y^2 - 10x^2y^3 - 5xy^4 \\ &= -5xy(x^3 + 2x^2y + 2xy^2 + y^3) \\ &= -5xy(x+y)(x^2 + xy + y^2) \\ &= 5(a-b)(b-c)(c-a)((a-b)^2 + (a-b)(b-c) + (b-c)^2) \\ &= 5(a-b)(b-c)(c-a)(a^2 + b^2 + c^2 - ab - bc - ca).\end{aligned}$$

Since a, b, c are distinct, we have $a^2 + b^2 + c^2 - ab - bc - ca = 0$, or $a^2 + b^2 + c^2 = 300$. Then $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) = 900$, so either $a + b + c = 30$ or $a + b + c = -30$.

First suppose $a + b + c = 30$, and suppose that $abc = k$ is a positive real number. Then a, b, c are solutions to the equation $x^3 - 30x^2 + 300x - k = 0$. We can rewrite the equation as

$$k - 1000 = x^3 - 30x^2 + 300x - 1000 = (x - 10)^3.$$

So if we let $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ be a third root of unity, then $a, b, c = 10 + \sqrt[3]{k-1000}, 10 + \omega\sqrt[3]{k-1000}, 10 + \omega^2\sqrt[3]{k-1000}$. Then a, b, c have integer real parts if $\sqrt[3]{k-1000}$ is an even integer. Since $\sqrt[3]{k-1000} > -10$, the first few possible values of k in this case are $1000 - 8^3 = 488, 1000 - 6^3 = 784, 1000 - 4^3 = 936, \ldots$

On the other hand, suppose $a+b+c = -30$, and suppose that $abc = k$ is a positive real number. Then a, b, c are solutions to the equation $x^3 + 30x^2 + 300x - k = 0$. We can rewrite the equation as

$$k + 1000 = x^3 + 30x^2 + 300x + 1000 = (x + 10)^3.$$

So if we let $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ be a third root of unity, then $a, b, c = -10 + \sqrt[3]{k+1000}, -10 + \omega\sqrt[3]{k+1000}, -10 + \omega^2\sqrt[3]{k+1000}$. Then a, b, c have integer real parts if $\sqrt[3]{k+1000}$ is an even integer. Since $\sqrt[3]{k+1000} > 10$, the first few possible positive values of k in this case are $12^3 - 1000 = 728, 14^3 - 1000 = 1744, \ldots$

We conclude that the three least possible positive real values of abc are 488, 728, and 784, and their sum is $\boxed{2000}$.

10. Complex numbers $a_1, a_2, \ldots, a_{10}$ satisfy the following property for each $1 \le k \le 10$:

$$\sum_{i=1}^{10} a_i^9 = 10a_k^{10} + a_k^9 - 2^9.$$

Given $a_1a_2 \cdots a_{10} = -\frac{m}{n}$ for relatively prime positive integers m and n, compute the sum of the digits of $m + n$.

Answer: 89

Solution: For each index k,

$$\frac{1}{10}\left(\sum_{i=1}^{10} a_i^9 - a_k^9 + 2^9\right) = a_k^{10}.$$

We can rearrange this as

$$a_k^{10} + \frac{1}{10}a_k^9 - \frac{1}{10}\left(\sum_{i=1}^{10} a_i^9 + 2^9\right) = 0.$$

Let $P_k = \sum_{i=1}^{10} a_i^k$ for each positive integer k. Then $a_1, a_2, \ldots, a_{10}$ are the roots of the polynomial $x^{10} + \frac{1}{10}x^9 - \frac{1}{10}(2^9 + P_9)$. Vieta's formulas give $P_1 = \sum_{i=1}^{10} a_i = -\frac{1}{10}$, $\prod_{i=1}^{10} a_i = -\frac{1}{10}(2^9 + P_9)$, and all other symmetric sums are 0. So, we can apply Newton's sums to evaluate $P_2, P_3, \ldots, P_9$:

$$\begin{aligned}
&P_2 = -\frac{1}{10}P_1 - 2(0) = \frac{1}{100}\\
&P_3 = -\frac{1}{10}P_2 - 0P_1 + 3(0) = -\frac{1}{1000}\\
&\ldots\\
&P_k = -\frac{1}{10}P_{k-1} = (-1)^k\frac{1}{10^k}\\
&\ldots\\
&P_9 = -\frac{1}{10}P_8 = -\frac{1}{10^9}
\end{aligned}$$

Finally,

$$\prod_{i=1}^{10} a_i = -\frac{1}{10}(2^9 + P_9) = \frac{-2^9 \cdot 10^9}{10^{10}} + \frac{1}{10^{10}} = -\frac{5119999999999}{10000000000}.$$

So, $m + n = 521999999999$, whose sum of digits is $5 + 2 + 1 + 9 \cdot 9 = \boxed{89}$.