1. Let $f(x) = x^2 - 11x + 31$. Distinct real numbers a and b satisfy $f(a) = f'(a)$ and $f(b) = f'(b)$. Given $a < b$, compute $10a + b$.

   Answer: 67

   Solution: The derivative of $f(x) = x^2 - 11x + 31$ is $f'(x) = 2x - 11$. Thus,

$$\begin{aligned} f(x) &= f'(x) \\ x^2 - 11x + 31 &= 2x - 11 \\ x^2 - 13x + 42 &= 0 \\ (x-6)(x-7) &= 0 \end{aligned}$$

   has solutions $x = 6$ and $x = 7$. Then $a = 6, b = 7$, and $10a + b = 60 + 7 = \boxed{67}$.

2. Determine the maximum value of

$$\frac{\left(2025^x\right)^\pi}{2025^{x^\pi}}$$

   over all positive real numbers x.

   Answer: $2025^{\pi-1}$

   Solution: Rewrite the expression as $2025^{h(x)}$, where $h(x) = \pi x - x^\pi$. Since 2025^t is increasing, it suffices to maximize $h(x)$ for $x > 0$.

$$h'(x) = \pi(1 - x^{\pi-1}), \qquad h''(x) = -\pi(\pi-1)x^{\pi-2} < 0.$$

$$h'(x) = \pi(1 - x^{\pi-1}) = 0 \implies 1 = x^{\pi-1} \implies x = 1.$$

   Thus for $x > 0$, h is concave down and has a unique critical point at $x = 1$, so its maximum is achieved at $x = 1$. The largest possible value of the original expression is

$$2025^{h(1)} = \boxed{2025^{\pi-1}}.$$

3. Let $f(x) = \sqrt{3x^2 + 2x + 1}$. Compute

$$\lim_{x\to\infty} f'(x).$$

   Answer: $\sqrt{3}$

   Solution: By the chain rule,

$$f'(x) = \frac{1}{2} \cdot \frac{6x+2}{\sqrt{3x^2+2x+1}} = \frac{3x+1}{\sqrt{3x^2+2x+1}} = \frac{3 + \frac{1}{x}}{\sqrt{3 + \frac{2}{x} + \frac{1}{x^2}}}.$$

   Taking $x \to \infty$ gives

$$\lim_{x\to\infty} f'(x) = \frac{3}{\sqrt{3}} = \boxed{\sqrt{3}}.$$

4. Compute

$$\lim_{n\to\infty} \left(\frac{2\tan^{-1}(n)}{\pi}\right)^n.$$

   Answer: $e^{-\frac{2}{\pi}}$

Solution:

Call the limit L. Taking the natural logarithm of the limit gives

$$\ln(L) = \ln\left(\lim_{n\to\infty}\left(\frac{2\tan^{-1}(n)}{\pi}\right)^n\right) = \lim_{n\to\infty}\ln\left(\left(\frac{2\tan^{-1}(n)}{\pi}\right)^n\right)$$

$$= \lim_{n\to\infty} n\ln\left(\frac{2\tan^{-1}(n)}{\pi}\right) = \lim_{n\to\infty}\frac{\ln\frac{2\tan^{-1}(n)}{\pi}}{1/n}.$$

Direct substitution gives an indeterminate form $\left(\frac{0}{0}\right)$ so L'Hôpital's rule can be applied. Differentiating the numerator and denominator separately gives

$$= \ln(L) = \lim_{n\to\infty}\frac{\frac{\pi}{2\tan^{-1}(n)}\cdot\frac{2}{\pi(1+n^2)}}{-\frac{1}{n^2}} = \lim_{n\to\infty} -\frac{n^2}{\tan^{-1}(n)(1+n^2)}$$

$$= \lim_{n\to\infty} -\frac{1}{\tan^{-1}(n)}\cdot\lim_{n\to\infty}\frac{n^2}{1+n^2} = -\frac{2}{\pi}\cdot 1.$$

Thus, $L = \boxed{e^{-\frac{2}{\pi}}}$.

5. For each real number n, let $\lfloor n\rfloor$ denote the greatest integer less than or equal to n, and let $\lceil n\rceil$ denote the least integer greater than or equal to n. Compute

$$\int_0^\infty \frac{(-1)^{\lfloor x\rfloor}}{\lceil x\rceil^2 - \lfloor x\rfloor^2}\,\mathrm{d}x\,.$$

Answer: $\boldsymbol{\frac{\pi}{4}}$

Solution: Note that

$$\begin{aligned}\int_0^\infty \frac{(-1)^{\lfloor x\rfloor}}{\lceil x\rceil^2 - \lfloor x\rfloor^2}\,\mathrm{d}x &= \sum_{i=0}^\infty \int_i^{i+1}\frac{(-1)^{\lfloor x\rfloor}}{\lceil x\rceil^2 - \lfloor x\rfloor^2}\,\mathrm{d}x\\ &= \sum_{i=0}^\infty \int_i^{i+1}\frac{(-1)^i}{(i+1)^2 - i^2}\,\mathrm{d}x\\ &= \sum_{i=0}^\infty \frac{(-1)^i}{i^2+2i+1-i^2}((i+1)-i)\\ &= \sum_{i=0}^\infty\frac{(-1)^i}{2i+1} = \arctan 1 = \boxed{\frac{\pi}{4}}\end{aligned}$$

by the Taylor series about $x=0$ of $\arctan(x) = \sum_{k=0}^\infty \frac{(-1)^k x^{2k+1}}{2k+1}$.

6. Evan wants to grow a garden, so he visits a florist selling n distinct flowers. He repeatedly rolls a fair n-sided die with faces labeled from 1 to n until he rolls a number other than 1. Evan then calculates N, the number of ways he can choose $\min\{k, n\}$ of the flowers, where k is the number of 1's rolled (possibly zero). Compute the limit as n goes to infinity of the expected value of N.

Answer: $\boldsymbol{e}$

Solution: For each $0 \leq k_0 \leq k$, the probability that $k = k_0$ is

$$\frac{1}{n^{k_0}}\left(1-\frac{1}{n}\right) = \frac{n-1}{n^{k_0+1}}.$$

Thus, we wish to calculate

$$\mathbb{E}[N] = \sum_{k_0=0}^{n} \binom{n}{k_0} P[\min\{k,n\} = k_0] \approx \sum_{k_0=0}^{n} \binom{n}{k_0} \frac{n-1}{n^{k_0+1}}$$

where we can ignore the case of rolling more than n 1's in a row because it quickly becomes negligible as $n \to \infty$. We then evaluate

$$\lim_{n\to\infty} \mathbb{E}[N] \approx \lim_{n\to\infty} \sum_{k_0=0}^{n} \binom{n}{k_0} \frac{n-1}{n^{k_0+1}} = \lim_{n\to\infty} \frac{n-1}{n} \sum_{k_0=0}^{n} \binom{n}{k_0} \frac{1}{n^{k_0}}$$

$$= \left(\lim_{n\to\infty} \frac{n-1}{n}\right) \cdot \left(\lim_{n\to\infty} \sum_{k_0=0}^{n} \binom{n}{k_0} \frac{1}{n^{k_0}}\right)$$

$$= (1) \cdot \left(\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n\right)$$

$$= 1 \cdot e = \boxed{e}$$

where the second-to-last equality follows from the Binomial theorem.

7. For positive real numbers a, let the circle $C(a)$ be the graph of $x^2+(y-a)^2 = 4a$ in the coordinate plane. Let $f(a)$ be the fraction of the area of $C(a)$ that lies above the parabola $y = x^2$. Find

$$\lim_{a\to\infty} f(a).$$

Answer: $\boldsymbol{\frac{\pi/3+\sqrt{3}/2}{\pi} = \frac{1}{3} + \frac{\sqrt{3}}{2\pi}}$**.**

Solution: Consider the change of variables $x = 2\sqrt{a}u$ and $y = a + 2\sqrt{a}v$ so that the circle becomes

$$u^2 + v^2 = 1$$

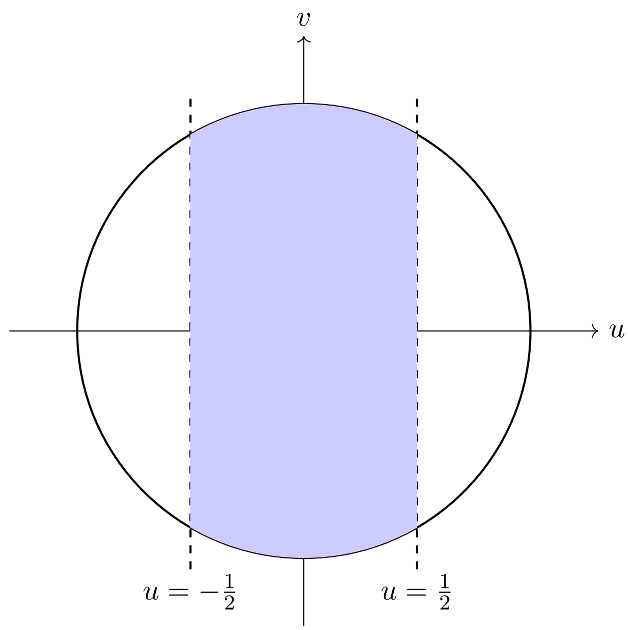
and the parabola becomes

$$a + 2\sqrt{a}v = 4au^2.$$

In the limit $a \to \infty$, the points on the parabola that intersect with the circle satisfy

$$a = 4au^2 \implies a(1-4u^2) = 0 \implies u = \pm\frac{1}{2}$$

since $|v| \leq 1$ on the circle, and $\sqrt{a} \ll a$ for large a. Then the sections of the parabola lying inside the circle approach vertical line segments at $u = \pm\frac{1}{2}$ as $a \to \infty$, as shown below.



Computing the fraction of area in the circle that lies above the parabola can be done by drawing radii to points of intersection, creating two circular sectors and two triangles. Each of the circular sectors measures 60° and so has area $\frac{\pi}{6}$, and each triangle has area $\frac{\sqrt{3}}{4}$. Thus, the fraction of the area of the circle lying above the parabola is

$$\frac{2\cdot\frac{\pi}{6}+2\cdot\frac{\sqrt{3}}{4}}{\pi}=\frac{\frac{\pi}{3}+\frac{\sqrt{3}}{2}}{\pi}=\boxed{\frac{1}{3}+\frac{\sqrt{3}}{2\pi}}.$$

8. An integer k is chosen uniformly at random from the set $\{0,1,\cdots,2^{2025}-1\}$. Compute the probability that

$$\int_0^\infty e^{-2025x}x^k\sin(2025x)\,\mathrm{d}x>0.$$

Answer: $\mathbf{\frac{3}{8}}$

Solution: First scale out the constant 2025 with $t=2025x$:

$$I_k:=\int_0^\infty e^{-2025x}\,x^k\sin(2025x)\,dx=\frac{1}{2025^{\,k+1}}\int_0^\infty e^{-t}\,t^k\sin t\,dt.$$

Thus the sign of I_k is the sign of

$$J_k:=\int_0^\infty e^{-t}\,t^k\sin t\,dt.$$

We use Euler's formula $\sin t=\mathrm{Im}(e^{it})$ and the following lemma.

Lemma. For any integer $k\geq 0$ and any complex number c with $\mathrm{Re}(c)>0$,

$$\int_0^\infty t^k e^{-ct}\,dt=\frac{k!}{c^{\,k+1}}.$$

Proof. For $k = 0$, $\int_0^\infty e^{-ct}\,dt = \frac{1}{c}$ since $\mathrm{Re}(c) > 0$. If it holds for $k-1$, then by integration by parts

$$\int_0^\infty t^k e^{-ct}\,dt = \Big[-\frac{t^k}{c}e^{-ct}\Big]_0^\infty + \frac{k}{c}\int_0^\infty t^{k-1}e^{-ct}\,dt = \frac{k}{c}\cdot\frac{(k-1)!}{c^k} = \frac{k!}{c^{k+1}},$$

since the boundary term vanishes when $\mathrm{Re}(c) > 0$. □

Applying the lemma with $c = 1 - i$ (note $\mathrm{Re}(1 - i) = 1 > 0$) and taking imaginary parts,

$$J_k = \mathrm{Im}\bigg(\int_0^\infty t^k e^{-(1-i)t}\,dt\bigg) = \mathrm{Im}\bigg(\frac{k!}{(1-i)^{k+1}}\bigg).$$

Write $1 - i = \sqrt{2}\,e^{-i\pi/4}$, so

$$(1-i)^{-(k+1)} = (\sqrt{2})^{-(k+1)}\,e^{i\,(k+1)\pi/4}.$$

Hence

$$J_k = k!\,(\sqrt{2})^{-(k+1)}\,\sin\bigg(\frac{(k+1)\pi}{4}\bigg).$$

Since the prefactor $k!\,(\sqrt{2})^{-(k+1)}$ is positive, the sign of J_k (and thus of I_k) is exactly the sign of $\sin\big(\frac{(k+1)\pi}{4}\big)$.

Now $\sin\big(\frac{(k+1)\pi}{4}\big)$ depends only on $k + 1 \pmod 8$:

$k+1 \pmod 8$	1	2	3	4	5	6	7	0
$\sin\big(\frac{(k+1)\pi}{4}\big)$	+	+	+	0	−	−	−	0

Equivalently,

$$I_k > 0 \iff k \equiv 0, 1, 2 \pmod 8, \qquad I_k = 0 \iff k \equiv 3, 7 \pmod 8, \qquad I_k < 0 \iff k \equiv 4, 5, 6 \pmod 8.$$

Finally, the 2^{2025} choices of k are evenly distributed among the 8 residue classes modulo 8. Exactly 3 of the 8 classes give a positive integral. Therefore,

$$\mathbb{P}\Big(I_k > 0\Big) = \boxed{\frac{3}{8}}.$$

9. Let $f(a, b) = (a^3 - 4a)(b - a) - \frac{(b-a)^2}{2}$. Find the least real number M such that for all finite, strictly increasing sequences $d_1, \ldots, d_n$ with $d_1 = -2$ and $d_n = 3$, the following inequality is satisfied:

$$\sum_{k=1}^{n-1} f(d_k, d_{k+1}) \le M.$$

Answer: 13

Solution: The sum $\sum_{k=1}^{n-1} f(d_k, d_{k+1})$ is a left-aligned Riemann sum of $\int_{-2}^{3}(x^3 - 4x)dx$, with a twist: the rectangles now have triangles chopped off from the top by lines of slope -1 starting from the left endpoint. If we took the standard limit of these sliced rectangles, these removed triangles become insignificant, and the result would just be the value of $\int_{-2}^{3}(x^3 - 4x)dx$.

We want to maximize the value of $\sum_{k=1}^{n-1} f(d_k, d_{k+1})$, so we want to not only cover the area under the graph $y = x^3 - 4x$, but also cover as much area above the curve as possible. Since the rectangles are cut off by lines with slope -1, it is best to start a rectangle wherever the slope transitions from greater than -1 to less than -1, because the line with slope -1 would be above the curve at that point.

Thus, we check when the derivative $3x^2-4$ is equal to -1, which is at $x=-1$ and $x=1$. The second derivative $6x$ is negative at $x=-1$ and positive at $x=1$, so we want to start a rectangle at $x=-1$. The line that cuts the rectangle starting at $x=-1$ is $y=-x+2$. It intersects the graph $y=x^3-4x$ at $x=-1$ and $x=2$, so we should stop the sliced rectangle at $x=2$.

Therefore, the sum $\sum_{k=1}^{n-1} f(d_k, d_{k+1})$ is optimized when we take the limit of the sliced rectangles over the interval $[-2,-1]$, use the sliced rectangle over the interval $[-1,2]$, and take the limit of the sliced rectangles over the interval $[2,3]$. So the least upper bound for the sum is:

$$\begin{aligned}\int_{-2}^{-1}(x^3-4x)dx+\frac{3\cdot 3}{2}+\int_2^3(x^3-4x)dx &= \frac{9}{2}+\left(\frac{x^4}{4}-2x^2\right)\Big|_{x=-2}^{x=-1}+\left(\frac{x^4}{4}-2x^2\right)\Big|_{x=2}^{x=3}\\ &= \frac{9}{2}+\frac{9}{4}+\frac{25}{4}\\ &= \boxed{13}.\end{aligned}$$

10. Compute

$$\int_0^{\frac{\pi}{2}} \ln\bigl(\sin^4(x)+\cos^4(x)\bigr)\,\mathrm{d}x\,.$$

Answer: $\boldsymbol{\frac{\pi}{2}\ln\left(\frac{3+2\sqrt{2}}{8}\right)}$ **OR** $\boldsymbol{\pi\ln\left(\frac{2+\sqrt{2}}{4}\right)}$

Solution: We can rewrite the integrand using $\sin^4(x)+\cos^4(x)=\frac{\tan^4(x)+1}{\sec^4(x)}$:

$$\int_0^{\frac{\pi}{2}} \ln\bigl(\sin^4(x)+\cos^4(x)\bigr)\mathrm{d}x=\int_0^{\frac{\pi}{2}}\ln\left(\frac{\tan^4(x)+1}{\sec^4(x)}\right)\mathrm{d}x=I+4K$$

where $I=\int_0^{\frac{\pi}{2}}\ln\bigl(1+\tan^4(x)\bigr)\,\mathrm{d}x$ and $K=\int_0^{\frac{\pi}{2}}\ln(\cos(x))\,\mathrm{d}x$.

First, we compute I. Substituting $u=\tan(x)$ gives

$$I=\int_0^\infty \frac{\ln\bigl(1+u^4\bigr)}{1+u^2}\mathrm{d}u=\int_0^\infty\frac{\ln\bigl(u^2-u\sqrt{2}+1\bigr)+\ln\bigl(u^2+u\sqrt{2}+1\bigr)}{1+u^2}\mathrm{d}u$$

by the factorization $1+u^4=(u^2-u\sqrt{2}+1)(u^2+u\sqrt{2}+1)$. Introduce a parameter α (Feynman's trick) and for $\alpha\ge 1$ define

$$I_1(\alpha)=\int_0^\infty\frac{\ln\bigl((1+u^2)\alpha-u\sqrt{2}\bigr)}{1+u^2}\,\mathrm{d}u$$

$$I_2(\alpha)=\int_0^\infty\frac{\ln\bigl((1+u^2)\alpha+u\sqrt{2}\bigr)}{1+u^2}\,\mathrm{d}u$$

so that $I=I_1(1)+I_2(1)$. Differentiating with respect to α and completing the square gives

$$I_1'(\alpha)=\frac{1}{\alpha}\int_0^\infty\frac{\mathrm{d}u}{u^2-\frac{u\sqrt{2}}{\alpha}+1}=\frac{1}{\alpha}\int_0^\infty\frac{\mathrm{d}u}{(u-\frac{\sqrt{2}}{2\alpha})^2+1-\frac{1}{2\alpha^2}}$$

$$I_2'(\alpha) = \frac{1}{\alpha}\int_0^\infty \frac{\mathrm{d}u}{u^2 + \frac{u\sqrt{2}}{\alpha} + 1} = \frac{1}{\alpha}\int_0^\infty \frac{\mathrm{d}u}{(u + \frac{\sqrt{2}}{2\alpha})^2 + 1 - \frac{1}{2\alpha^2}}$$

Motivated by the completed squares in the denominators, let $u = \frac{\sqrt{2}}{2\alpha} + \tan(\theta)\sqrt{1 - \frac{1}{2\alpha^2}}$ for I_1' and $u = -\frac{\sqrt{2}}{2\alpha} + \tan(\theta)\sqrt{1 - \frac{1}{2\alpha^2}}$ for I_2'. Then

$$I_1'(\alpha) = \frac{\sqrt{2}}{\sqrt{2\alpha^2 - 1}}\left(\frac{\pi}{2} + \arctan\left(\frac{1}{\sqrt{2\alpha^2-1}}\right)\right)$$
$$I_2'(\alpha) = \frac{\sqrt{2}}{\sqrt{2\alpha^2 - 1}}\left(\frac{\pi}{2} - \arctan\left(\frac{1}{\sqrt{2\alpha^2-1}}\right)\right).$$

We have

$$(I_1 + I_2)'(\alpha) = \frac{2\pi}{\sqrt{4\alpha^2 - 2}}.$$

Integrating via $\alpha = \frac{1}{\sqrt{2}}\sec(\theta)$ gives

$$(I_1 + I_2)(\alpha) = \pi \ln\left(\alpha\sqrt{2} + \sqrt{2\alpha^2 - 1}\right) + C.$$

To find the constant C, we evaluate $(I_1 + I_2)(\sqrt{2})$ by splitting $\ln\left(2u^4 + 2u^2 + 2\right) = \ln(2) + \ln\left(u^4 + u^2 + 1\right)$:

$$(I_1 + I_2)(\sqrt{2}) = \underbrace{\int_0^\infty \frac{\ln(2)}{1+u^2}\,\mathrm{d}u}_{\frac{\pi}{2}\ln(2)} + \int_0^\infty \frac{\ln\left(u^4 + u^2 + 1\right)}{1+u^2}\,\mathrm{d}u.$$

The second term can be evaluated using another Feynman step.

Let

$$J(\alpha) = \int_0^\infty \frac{\ln\left(\alpha^2 u^2 (u^2+1) + 1\right)}{u^2 + 1}\,\mathrm{d}u.$$

For $\alpha > 0$, differentiating with respect to α yields

$$\begin{aligned} J'(\alpha) &= \frac{1}{\alpha}\int_0^\infty \frac{2\,\mathrm{d}u}{\left(u - \frac{1}{\alpha u}\right)^2 + \frac{2}{\alpha} + 1} \\ &= \frac{1}{\alpha}\int_0^\infty \frac{2\,\mathrm{d}u}{u^2 + \frac{2}{\alpha} + 1}, \end{aligned}$$

where the last equality follows by the Cauchy-Schlömilch transformation. Evaluating the integral gives $J'(\alpha) = \dfrac{\pi}{\sqrt{\alpha(\alpha+2)}}$. Integrating and using $J(0) = 0$ by continuity shows that

$$J(\alpha) = \pi \ln\left(\alpha + 1 + \sqrt{\alpha^2 + 2\alpha}\right).$$

In particular, $J(1) = \pi \ln\left(2 + \sqrt{3}\right)$, and we can now compute C:

$$\begin{aligned} C &= (I_1 + I_2)(\sqrt{2}) - \pi \ln\left(2 + \sqrt{3}\right) \\ &= \frac{\pi}{2}\ln(2) + \pi \ln\left(2 + \sqrt{3}\right) - \pi \ln\left(2 + \sqrt{3}\right) \\ &= \frac{\pi}{2}\ln(2) \end{aligned}$$

Finally,

$$I = (I_1 + I_2)(1) = \pi \ln\left(\sqrt{2} + 1\right) + \frac{\pi}{2}\ln(2).$$

Now we compute K. Since $\sin(x) = \cos\left(\frac{\pi}{2} - x\right)$, we can make the substitution $u = \frac{\pi}{2} - x$. Then we can write $K = \int_0^{\frac{\pi}{2}} \ln(\cos(x))\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln(\sin(x))\,\mathrm{d}x$. Adding both representations and using $\sin(2x) = 2\sin(x)\cos(x)$ gives

$$2K = \int_0^{\frac{\pi}{2}} \ln\left(\frac{\sin(2x)}{2}\right)\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln(\sin(2x))\,\mathrm{d}x - \frac{\pi}{2}\ln(2).$$

Substituting $t = 2x$,

$$\int_0^{\frac{\pi}{2}} \ln(\sin(2x))\,\mathrm{d}x = \frac{1}{2}\int_0^{\pi} \ln(\sin(t))\,\mathrm{d}t = \frac{1}{2}\cdot 2\int_0^{\frac{\pi}{2}} \ln(\sin(t))\,\mathrm{d}t = K.$$

We have $K = -\dfrac{\pi}{2}\ln(2)$ and $4K = -2\pi\ln(2)$ and can now finish the problem:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \ln\left(\sin^4(x) + \cos^4(x)\right)\mathrm{d}x &= \pi \ln\left(\sqrt{2} + 1\right) + \frac{\pi}{2}\ln(2) - 2\pi\ln(2) \\ &= \boxed{\frac{\pi}{2}\ln\left(\frac{3 + 2\sqrt{2}}{8}\right)} \end{aligned}$$