1. Let $f(x) = \frac{x+1}{x^3+1}$. Find the real number c such that

$$f(c) = \lim_{x \to -1} f(x).$$

Answer: 2

Solution: We have that

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x+1}{x^3+1} = \lim_{x \to -1} \frac{1}{x^2-x+1} = \frac{1}{3},$$

so we need to find c such that $f(c) = \frac{1}{3}$. We have

$$\frac{1}{c^2-c+1} = \frac{1}{3} \implies c^2 - c + 1 = 3,$$

so $c = -1$ or $c = 2$. However, $c = -1$ is impossible since $f(-1)$ is undefined. Thus, $c = \boxed{2}$.

2. For positive real numbers a and b, define the function

$$f(a,b) = \int_0^{a/b} \frac{1}{a+bx}\,\mathrm{d}x.$$

Compute $\max\{f(20,25), f(25,20)\}$.

Answer: $\boldsymbol{\dfrac{\ln(2)}{20}}$

Solution: For positive a, b,

$$\begin{aligned} f(a,b) &= \int_0^{a/b} \frac{1}{a+bx}\,\mathrm{d}x \\ &= \frac{1}{b}\left[\ln(a+bx)\right]_0^{a/b} \\ &= \frac{\ln 2}{b}. \end{aligned}$$

Therefore

$$f(20,25) = \frac{\ln 2}{25}, \qquad f(25,20) = \frac{\ln 2}{20},$$

so the maximum is $\boxed{\dfrac{\ln 2}{20}}$.

3. Compute

$$\int_0^1 \sum_{n=1}^{\infty} \sin\left(\frac{\pi}{4}(2n-1)\right)\frac{x^n}{n}\,\mathrm{d}x.$$

Answer: $\boldsymbol{\dfrac{\pi-2}{2\sqrt{2}}}$

Solution: First, handle the interior sum by splitting it into even and odd terms:

$$\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{4}(2n-1)\right)\frac{x^n}{n} = \sum_{i=1}^{\infty} \sin\left(\frac{\pi}{4}(4i-3)\right)\frac{x^{2i-1}}{2i-1} + \sum_{j=1}^{\infty} \sin\left(\frac{\pi}{4}(4j-1)\right)\frac{x^{2j}}{2j}$$

The odd-term sum is the Maclaurin series of $\frac{1}{\sqrt{2}}\tan^{-1}x$:

$$\sum_{i=1}^{\infty}\sin\left(\frac{\pi}{4}(4i-3)\right)\frac{x^{2i-1}}{2i-1}=\frac{1}{\sqrt{2}}\left(\frac{x^1}{1}-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}\dots\right)=\frac{1}{\sqrt{2}}\tan^{-1}x$$

and the even-term sum is the Maclaurin series of $\frac{1}{2\sqrt{2}}\ln\left(1+x^2\right)$:

$$\sum_{j=1}^{\infty}\sin\left(\frac{\pi}{4}(4j-1)\right)\frac{x^{2j}}{2j}=\frac{1}{\sqrt{2}}\left(\frac{x^2}{2}-\frac{x^4}{4}+\frac{x^6}{6}-\frac{x^8}{8}\dots\right)=\frac{1}{2\sqrt{2}}\ln\left(1+x^2\right).$$

The desired integral is

$$\frac{1}{\sqrt{2}}\int_0^1\left(\tan^{-1}x+\frac{1}{2}\ln\big(1+x^2\big)\right)\mathrm{d}x.$$

From

$$\int_0^1\tan^{-1}x\,\mathrm{d}x=\frac{\pi}{4}-\frac{1}{2}\ln 2$$

and

$$\int_0^1\ln\big(1+x^2\big)\,\mathrm{d}x=\ln 2-2+\frac{\pi}{2},$$

the final answer is

$$\frac{1}{\sqrt{2}}\left(\frac{\pi}{2}-1\right)=\boxed{\frac{\pi-2}{2\sqrt{2}}}.$$