1. Jordan has two white socks and two black socks in their sock drawer. If they pull out two socks uniformly at random without replacement, what is the probability that they are the same color?

   Answer: $\frac{1}{3}$

   Solution: After the first sock is pulled out, there are three other socks in the drawer. Only one of the socks left in the drawer is the same color as the first sock that was taken. Thus, the probability that the second sock matches the first is $\boxed{\frac{1}{3}}$.

2. Find the positive integer n such that $n+10$ is both a multiple of n and a factor of $n+100$.

   Answer: 5

   Solution: Note that if n and x are integers such that $n+x$ is a multiple of n, then x is a multiple of n. In particular, if $n+x=ny$ for some integer y, then $x=ny-n=n(y-1)$, which is a multiple of n.

   Thus, if $n+10$ is a multiple of n, then 10 is a multiple of n, so $n=1,2,5$, or 10.

   Similarly, if $n+10$ is a factor of $n+100=(n+10)+90$, then 90 is a multiple of $(n+10)$. Trying $n=1,2,5,10$ gives $n+10=11,12,15,20$, of which only 15 divides 90. Thus, $n=\boxed{5}$.

3. In how many ways can ten identical white blocks be placed into a red box, a yellow box, a green box, and a blue box such that no box is empty, and at least two boxes contain the same number of blocks?

   Answer: 60

   Solution 1: List out the different arrangements up to permutation, taking care not to double-count:

   $$\{1,1,1,7\},\{1,1,2,6\},\{1,1,3,5\},\{1,1,4,4\},\{2,2,1,5\},\{2,2,2,4\},\{3,3,2,2\},\{3,3,3,1\}.$$

   There are three cases where three boxes have the same number of blocks in them, which can happen $\frac{4!}{3!1!}=4$ ways each. There are two cases where there are two pairs of boxes with an equal number of blocks between them, which can happen $\frac{4!}{2!2!}=6$ different ways each. Lastly, there are three cases where two boxes have the same number of blocks, and the remaining two boxes have distinct numbers of blocks. This can happen $\frac{4!}{2!1!1!}=12$ different ways for each of the three cases. Thus, there are $(3\cdot4)+(2\cdot6)+(3\cdot12)=\boxed{60}$ arrangements.

   Solution 2: If we ignore the condition that two boxes have the same number of blocks, we can use stars and bars to show that there are $\binom{10-1}{4-1}=\binom{9}{3}=84$ distinct ways to distribute ten identical blocks among four different boxes. The only case where the number of blocks in each of the four boxes is distinct is $\{1,2,3,4\}$, and there are $4!=24$ ways to arrange this. Thus, the number of arrangements that satisfy the given conditions is $84-24=\boxed{60}$.

4. A *substring* of a positive integer n is a nonnegative integer whose digits appear consecutively in the digits of n. For example, 10, $00=0$, and $02=2$ are two-digit substrings of 1002, but 12 and $01=1$ are not. An integer n is chosen uniformly at random from the integers between 1000 and 9999 inclusive. Compute the probability that all two-digit substrings of n are divisible by three.

   Answer: $\frac{59}{1500}$

   Solution: There are $9999-1000+1=9000$ integers that could be chosen, each with equal probability. We proceed by counting the number of these integers n that have all two-digit substrings divisible by 3. Write n as $\overline{d_3d_2d_1d_0}$, where we allow digits other than d_3 to be zero.

For $k = 0, 1,$ or 2, the number $\overline{d_{k+1}d_k}$ must be divisible by three, which is equivalent to requiring $d_k + d_{k+1}$ to be divisible by 3. We now have the following cases.

- Suppose d_0 is divisible by 3. Then all digits are divisible by 3, so each has four options among $\{0, 3, 6, 9\}$, except d_3 must be nonzero. Thus, there are $3 \cdot 4^3 = 3 \cdot 64 = 192$ positive integers here.
- Suppose d_0 is not divisible by 3. Then d_0 and d_2 leave the same remainder when divided by 3, and likewise for d_1 and d_3; additionally, $d_0 + d_1$ is divisible by 3. There are 6 options for d_0, and then each remaining digit has 3 options, so there are $6 \cdot 3^3 = 6 \cdot 27 = 162$ positive integers here.

There are $192 + 162 = 354$ values of n which meet our criteria. The probability of choosing one of these values is $\frac{354}{9000} = \boxed{\dfrac{59}{1500}}$.

5. Over all triples of positive integers (a, b, c) satisfying the equation

$$\gcd\left(\operatorname{lcm}(a, b), \operatorname{lcm}(a, c), \operatorname{lcm}(b, c)\right) = 2025,$$

compute the smallest possible value of $a + b + c$.

Answer: 2131

Solution: Note that $2025 = 3^4 \cdot 5^2$ divides $\operatorname{lcm}(a, b)$, $\operatorname{lcm}(a, c)$, and $\operatorname{lcm}(b, c)$. Thus in each pair (a, b), (a, c) and (b, c), at least one member of the pair must be divisible by 3^4 and at least one member of the pair must be divisible by 5^2.

By the pigeonhole principle, at least one of a, b, and c is divisible by both 3^4 and 5^2. Without loss of generality, let c be divisible by 3^4 and 5^2, so $c \geq 3^4 \cdot 5^2 = 2025$.

At least one of a and b must be divisible by 3^4, and at least one must be divisible by 5^2. There are two cases: at least one of a, b is divisible by both 3^4 and 5^2, or one is divisible by 3^4 but not 5^2, and the other is divisible by 5^2 but not 3^4.

The second case yields a and b which have a smaller sum: 81 and 25 (the other case gives a minimum value of 2025 for one of a or b).

Thus, the smallest possible value of $a + b + c$ is $81 + 25 + 2025 = \boxed{2131}$.

6. An increasing sequence of positive integers is *three-hating* if it contains at least one term and alternates between integers that are multiples of 3 and integers that are not multiples of 3. For example, $(1), (3, 4, 6)$, and (3) are three-hating while $(3, 6)$ and $(3, 6, 7, 9)$ are not. Compute the number of three-hating sequences in which every term is less than or equal to 12.

Answer: 360

Solution: For each positive integer N, let $f(N)$ denote the number of *three-hating* sequences ending with N. We can split into two cases:

Case 1: N is divisible by 3. Then the sequence excluding the last term ends with a number less than N which is not divisible by 3, or is empty.

Case 2: N is not divisible by 3. Then the sequence excluding the last term ends with a number less than N which is divisible by 3, or is empty.

These cases give the recursions

$$f(N) = 1 + \sum_{n<N, 3\nmid n} f(n) \ \text{ if } N \text{ is divisible by 3,}$$

$$f(N) = 1 + \sum_{n<N, 3\mid n} f(n) \ \text{ if } N \text{ is not divisible by 3.}$$

It is straightforward to compute $f(1) = 1, f(2) = 1, f(3) = 3$. We can then build the following table using the recursive relations above:

N	$f(N)$
1	1
2	1
3	3
4	4
5	4
6	11
7	15
8	15
9	41
10	56
11	56
12	153

We want to include all sequences ending with a number less than or equal to 12, so we add the values in the $f(N)$ column and get $\boxed{360}$.

7. Fourteen people are sitting at a circular table with fourteen distinct seats. They all get up to eat lunch and then return to the table. When they sit down again, each person sits either in their original seat or a seat adjacent to their original seat, and no two people sit in the same seat. In how many ways can this occur?

   Answer: 845

   Solution: Let $N = 14$ for convenience.

   Assign a number to each person and each chair so that person 1 originally sat in chair 1, person 2 originally sat in the next chair going clockwise, and so on. Consider person 1's new seat. They have three options: they can sit back in chair 1, or they can sit in one of the two adjacent chairs $(2, N)$, which are equivalent by symmetry. If person 1 sits in their original chair, then the other $N - 1$ chairs are effectively in a row (as chairs 2 and N now only have 1 neighbor, equivalent to a line of chairs rather than a circle).

   Thus, we compute the number of ways for every person to be in their original seat or in an adjacent seat for a line of m chairs as a subproblem. Call the answer to this subproblem l_m. The person who sits back down in the first chair must either be the person who was originally in the first chair or the person who was originally in the second chair. In the first case, we have reduced the problem to a line of $m - 1$ chairs, which can be sat in l_{m-1} ways. In the second case, the person who was originally in the first chair must now sit in the second chair, and so we have reduced the problem to a line of $m - 2$ chairs, which can be sat in l_{m-2} ways. Thus,

$l_m = l_{m-1} + l_{m-2}$. We can compute $l_0 = l_1 = 1$, so $l_m = F_{m+1}$ where F_k is the kth Fibonacci number.

Now, we return to the original problem. We have handled the case where person 1 returns to chair 1, where there are $l_{N-1} = F_N$ ways of seating everyone. If person 1 sits in chair 2 (or chair N by symmetry), there are two possibilities for where person 2 can sit. One possibility is that person 2 sits in chair 1, and we again have a line of $N-2$ chairs with F_{N-1} ways for the people to sit. The other possibility is that person 2 sits in chair 3, and there is a forced cycle: person 3 must sit in chair 4, and so on, with person k sitting in chair $k+1$ until we reach $k = N$ and wrap around to chair 1. Therefore, if person 1 sits in chair 2, there are $F_{N-1}+1$ ways to seat the remainder of the table. By symmetry, there is the same number of ways to seat the table if person 1 sits in chair N, so our final answer is $F_N + 2(F_{N-1}+1)$. Substituting $N = 14$ and computing the Fibonacci sequence yields $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, F_{13} = 233, F_{14} = 377$. The final answer is $377 + 2(233+1) = \boxed{845}$.

8. Let $N = 2025 \times 202^5 \times 20^{25}$. For each positive integer x, define $\sigma(x)$ to be the sum of the positive divisors of x. Define $h(x)$ to be the sum of all positive divisors y of x such that $\sigma(y)$ is even. For example, $\sigma(9) = 13$ and $h(9) = 3$. Find the number of positive divisors k of N such that $h(k)$ is odd.

   Answer: 13328

   Solution: Let's first understand when $\sigma(x)$ is even; to do that, we find when $\sigma(x)$ is odd. Fix $x = 2^k p_1^{r_1} \cdots p_n^{r_n}$. Let $P_x = \{p_1, \dots, p_n\}$ be the set of the odd primes dividing x. Then

$$\sigma(x) = \left(\sum_{\substack{i\geq 0\\ 2^i \mid x}} 2^i\right) \prod_{p\in P_x} \left(\sum_{\substack{i\geq 0\\ p^i \mid x}} p^i\right)$$

$$\sigma(x) = (1 + \cdots + 2^k)(1 + p_1 + \cdots + p_1^{r_1}) \dots (1 + p_n + \cdots + p_n^{r_n})$$

   Note that $\sigma(x)$ is odd if and only if every factor in the product above is odd. The factor $\left(\sum_{\substack{i\geq 0\\ 2^i \mid x}} 2^i\right)$ is always odd; however, for any odd prime, there must be an odd number of terms in $\left(\sum_{\substack{i\geq 0\\ p^i \mid x}} p^i\right)$ in order for $\sigma(x)$ to be odd. Thus, the odd part of x must be a perfect square (i.e. x must be a perfect square or twice a perfect square) in order for $\sigma(x)$ to be odd, and $\sigma(x)$ is even otherwise.

Now, let's consider the parity of $h(x)$. Note that

$$\begin{aligned} h(x) &= \sigma(x) - \sum_{\substack{y \mid x \\ 2 \nmid \sigma(y)}} y \\ &= \left(\sum_{\substack{i \geq 0 \\ 2^i \mid x}} 2^i \right) \prod_{p \in P_x} \left(\sum_{\substack{i \geq 0 \\ p^i \mid x}} p^i \right) - \left(\sum_{\substack{i \geq 0 \\ 2^i \mid x}} 2^i \right) \prod_{p \in P_x} \left(\sum_{\substack{i \geq 0 \\ p^{2i} \mid x}} p^{2i} \right) \\ &= \left(\sum_{\substack{i \geq 0 \\ 2^i \mid x}} 2^i \right) \left(\prod_{p \in P_x} \left(\sum_{\substack{i \geq 0 \\ p^i \mid x}} p^i \right) - \prod_{p \in P_x} \left(\sum_{\substack{i \geq 0 \\ p^{2i} \mid x}} p^{2i} \right) \right). \end{aligned}$$

As stated before, $\left(\sum_{\substack{i \geq 0 \\ 2^i \mid x}} 2^i \right)$ is always odd, so we consider the parities of $\prod_{p \in P_x} \left(\sum_{\substack{i \geq 0 \\ p^i \mid x}} p^i \right)$ and $\prod_{p \in P_x} \left(\sum_{\substack{i \geq 0 \\ p^{2i} \mid x}} p^{2i} \right)$. Define $a(x) = \prod_{p \in P_x} \left(\sum_{\substack{i \geq 0 \\ p^i \mid x}} p^i \right)$ and $b(x) = \prod_{p \in P_x} \left(\sum_{\substack{i \geq 0 \\ p^{2i} \mid x}} p^{2i} \right)$. As before, $a(x)$ is odd when the odd part of x is a perfect square, and $b(x)$ is odd if $r_i \equiv 0$ or $1 \pmod 4$ for all i.

We consider two cases to determine the number of the factors $x = 2^k 3^{r_1} 5^{r_2} 101^{r_3}$ of $N = 2^{55} 3^4 5^{27} 101^5$ such that $h(x)$ is odd:

- $a(x)$ is odd, $b(x)$ is even. Then for all i, $r_i \equiv 0 \pmod 2$, and there is at least one $r_i \equiv 2, 3 \pmod 4$. By complementary counting, the number of factors of N satisfying this is

$$(56)(3)(14)(3) - (56)(2)(7)(2)$$

- $a(x)$ is even, $b(x)$ is odd. Then for all i, $r_i \equiv 0, 1 \pmod 4$, and there is at least one $r_i \equiv 1 \pmod 2$. By complementary counting, the number of factors of N satisfying this is

$$(56)(3)(14)(4) - (56)(2)(7)(2)$$

Thus, the answer is

$$(56)(3)(14)(3) + (56)(3)(14)(4) - 2(56)(2)(7)(2) = 56(294 - 56) = \boxed{13328}.$$

9. In a tournament, twelve players compete in a sequence of matches. Each match consists of two players, and no two matches have the same pair of players. Furthermore, no two players who compete in the same number of matches in the tournament compete against each other in a match. What is the greatest possible number of matches that can be played in the tournament?

Answer: 51

Solution: Represent the tournament as a graph G, where players are vertices and matches are edges: two vertices are connected by an edge if and only if the players corresponding to the vertices played a match against each other. Then G is a graph with 12 vertices such that no two vertices of the same degree are adjacent.

Here is some relevant terminology:

- The *complete graph* K_n is a graph of n vertices where each pair of vertices is connected by an edge.
- For any graph H, denote $d_H(v)$ to be the number of edges with v as an endpoint, or the *degree* of v, in H.
- The *complement* of graph G, denoted $\overline{G}$, is the graph with the same vertices as G, and two vertices v and w are connected by an edge in $\overline{G}$ if and only if they are *not* connected by an edge in G.
- A *connected component* of a graph is a subgraph H' of a graph H in which there exists a path between any two vertices in H', but there are no paths connecting vertices in H' and vertices not in H'. (An isolated vertex with no edges connecting it to any other vertex is a connected component.)

First, $d_G(v) = 11 - d_{\overline{G}}(v)$ for all $v \in G$. Since the total number of edges in G and $\overline{G}$ is $\binom{12}{2} = 66$, we can alternatively minimize the number of edges in the graph of $\overline{G}$.

If v and w have the same degree, then there is not an edge in G connecting v and w, so v and w are adjacent in $\overline{G}$. Thus, for each nonnegative integer k, all of the vertices of $\overline{G}$ with degree k are adjacent to each other. This means that there are at most $k+1$ vertices with degree k in $\overline{G}$. Since $12 = 1+2+3+4+2$, the sum of the degrees in $\overline{G}$ that can possibly be achieved by picking the least possible degree measures is at least

$$1 \cdot 0 + 2 \cdot 1 + 3 \cdot 2 + 4 \cdot 3 + 2 \cdot 4 = 28.$$

Then the number of edges in $\overline{G}$ is at least $\frac{1}{2} \cdot 28 = 14$. We now demonstrate that $\overline{G}$ cannot have 14 edges. The only way for the sum of degrees of vertices of $\overline{G}$ to be 28 is if there are $k+1$ vertices with degree k for each $k = 0, 1, 2, 3$ and 2 vertices with degree 4. However, that means that each group of $k+1$ vertices with degree k is a complete graph and connected component, as no other edges can be attached to them. This leaves 2 vertices with degree 4 that cannot be connected by an edge to any other vertices except each other, so the degree of these vertices is at most 1, contradiction. Therefore, $\overline{G}$ has at least 15 edges.

Now, we show that it is possible for $\overline{G}$ to have exactly 15 edges. In order to fix the issue that arises with the 2 vertices with degree 4, we can increase 2 of the 4 vertices with degree 3 so that there are instead 2 vertices with degree 3 and 4 vertices with degree 4. Then, motivated by complete graphs, we can construct the desired graph $\overline{G}$ as follows:

- Let $\overline{G}$ have 4 connected components consisting of K_1, K_2, K_3, and another graph H.
- To construct graph H, take a copy of K_2 with vertices labeled a_1, a_2 and a copy of K_4 with vertices labeled b_1, b_2, b_3, b_4. Then draw the edges $a_1b_1, a_1b_2, a_2b_3, a_2b_4$. Then a_1, a_2 have degree 3 and b_1, b_2, b_3, b_4 have degree 4.

Each pair of vertices with the same degree is connected by an edge in $\overline{G}$ in this construction. Therefore, we have found an example where $\overline{G}$ has exactly 15 edges. We conclude that the least possible number of edges in $\overline{G}$ is 15, so the greatest possible number of edges in G is $66 - 15 = \boxed{51}$.

10. Let S be the set of triples of nonnegative integers (p, q, r) such that

$$3^p \cdot 21^q + 9 \cdot 2^r$$

is a perfect square. Compute

$$\sum_{(p,q,r)\in S} 100p + 10q + r.$$

Answer: 1998

Solution: We want to solve the Diophantine equation $3^p \cdot 21^q + 9 \cdot 2^r = d^2$. We simplify the expression:

$$3^p \cdot 21^q + 9 \cdot 2^r = 3^{p+q} \cdot 7^q + 3^2 \cdot 2^r = 3^2(3^{p+q-2} \cdot 7^q + 2^r).$$

If $p+q \geq 2$, the expression is a perfect square if and only if $3^{p+q-2} \cdot 7^q + 2^r$ is a perfect square, so we dispense with the smaller cases. If $p+q = 1$, then

$$3^{p+q} \cdot 7^q + 3^2 \cdot 2^r = 3(7^q + 3 \cdot 2^r)$$

has exactly one factor of 3, and is thus not a perfect square.

If $p+q = 0$, it follows that $p = 0$ and $q = 0$, so

$$3^{p+q} \cdot 7^q + 3^2 \cdot 2^r = 1 + 9 \cdot 2^r.$$

If r were even, this expression would be one more than a positive perfect square, so it would not be a perfect square. So assume r is odd; then $r = 2s+1$ for some nonnegative integer s. Rearranging and factoring,

$$18 \cdot 2^{2s} = d^2 - 1 = (d+1)(d-1).$$

Since $d+1$ and $d-1$ are not both multiples of 3, and since $\gcd(d-1, d+1) = 2$, $d+1$ and $d-1$ are $9 \cdot 2^t$ and 2^u in some order, for some positive integers t and u where

$$t + u = 2s + 1.$$

Then $9 \cdot 2^t - 2^u$ is equal to 2 or -2, which has exactly one factor of 2, so either $t = 1$ or $u = 1$. If $t = 1$, then

$$|18 - 2^u| = 2,$$

which gives $u = 4$. If $u = 1$, then

$$|9 \cdot 2^t - 2| = 2,$$

which is impossible. Thus, $t = 1$ and $u = 4$ is the only solution, giving $s = 2$ and $r = 5$. This gives $(p,q,r) = (0,0,5)$ as a solution.

Now assume $p+q \geq 2$. We can then reframe the question: let $a = p+q-2$, $b = q$, and $c = r$. Note that $p = a - b + 2$. Then we want to find the solutions to the Diophantine equation

$$3^a 7^b + 2^c = d^2$$

where a, b, and c are nonnegative integers such that $a \geq b - 2$. First, we handle the cases where $c \leq 2$.

If $c = 0$, then

$$3^a 7^b = d^2 - 1 = (d-1)(d+1).$$

Since d is even, $\gcd(d-1, d+1) = 1$. Thus, $d-1$ and $d+1$ cannot both be multiples of 3, and cannot both be multiples of 7, so either one of them is 1 and the other is $3^a 7^b$, or one of them is 3^a and the other is 7^b. If one of them is 1, then the other is 3, which is already encapsulated in the latter case. Thus, either

$$3^a - 7^b = 2$$

or

$$3^a - 7^b = -2.$$

If $a = 0$, the second equation would give $7^b = 3$, which is impossible. If $a \geq 1$, then since $7 \equiv 1 \pmod 3$, taking $3^a - 7^b = -2$ modulo 3 gives

$$-1 \equiv -2 \pmod 3,$$

which is not possible. Therefore,

$$3^a - 7^b = 2.$$

If $b = 0$, then $3^a = 3$, giving $a = 1$. If $b = 1$, then $3^a = 9$, giving $a = 2$. Thus, we get $(a, b, c) = (1, 0, 0), (2, 1, 0)$ as solutions for this case.

To prove there are no other solutions to $3^a - 7^b = 2$, suppose that $b \geq 2$. Then $3^a = 7^b + 2 \geq 51$, so $a \geq 4$. We can rewrite

$$3^a - 9 = 7^b - 7,$$

or

$$9(3^{a-2} - 1) = 7(7^{b-1} - 1),$$

and then let $m = a - 2$ and $n = b - 1$. Then $7 \mid 3^m - 1$, so $6 \mid m$, and hence

$$3^6 - 1 \mid 3^m - 1.$$

Since $3^6 - 1$ has a factor of $3^2 + 3 + 1 = 13$, we have $13 \mid 7^n - 1$, so $12 \mid n$. Thus,

$$7^{12} - 1 \mid 7^n - 1.$$

Since $7^{12} - 1$ has a factor of $7^2 - 7 + 1 = 43$, we have $43 \mid 3^m - 1$, and so $42 \mid m$. Then

$$3^{42} - 1 \mid 3^m - 1.$$

Since $49 \mid 3^{42} - 1$ by Euler's totient theorem, we have

$$49 \mid 3^m - 1.$$

It follows from

$$9(3^m - 1) = 7(7^n - 1)$$

that $49 \mid 7(7^n - 1)$. However, $7 \nmid 7^n - 1$, so the right-hand side has exactly one factor of 7, which is a contradiction. This completes the case $c = 0$.

If $c = 1$, then

$$3^a 7^b = d^2 - 2.$$

Since $d^2 - 2 \equiv 1, 2 \pmod 3$, it is not divisible by 3, so we must have $a = 0$. Since $a \geq b - 2$, we also have $b \leq 2$. Trying these out, $3^0 \cdot 7^0 + 2 = 3$ and $3^0 \cdot 7^2 + 2 = 51$ are not perfect squares, but $3^0 \cdot 7^1 + 2 = 9$ is, yielding the triple $(a, b, c) = (0, 1, 1)$ for this case.

If $c = 2$, then

$$3^a 7^b = d^2 - 4 = (d - 2)(d + 2).$$

Since d is odd, $\gcd(d-2, d+2) = 1$. Thus, $d-2$ and $d+2$ cannot both be multiples of 3, and cannot both be multiples of 7. If one factor were 1 and the other were $3^a 7^b$, then the other factor would be 5, which is impossible. Therefore, one factor is 3^a and the other is 7^b. Thus, either

$$3^a - 7^b = 4$$

or

$$3^a - 7^b = -4.$$

If $a = 0$, then $3^a - 7^b \leq 0$, so the first equation is impossible. If $a \geq 1$, then since $7 \equiv 1 \pmod 3$, taking $3^a - 7^b = 4$ modulo 3 gives

$$-1 \equiv 4 \pmod 3,$$

which is not possible. Therefore,

$$3^a - 7^b = -4.$$

Checking $a \leq 3$ or $b \leq 1$, we get $a = 1$, $b = 1$ as the only solution, giving $(a, b, c) = (1, 1, 2)$ as a triple in this case.

To prove there are no other solutions to $3^a - 7^b = -4$, suppose that $b \geq 2$ and $a \geq 4$. Then we can rewrite

$$3^a - 3 = 7^b - 7,$$

or

$$3(3^{a-1} - 1) = 7(7^{b-1} - 1).$$

Let $m = a - 1$ and $n = b - 1$. Then $7 \mid 3^m - 1$, so $6 \mid m$, and hence

$$3^6 - 1 \mid 3^m - 1.$$

Since $13 \mid 3^6 - 1$, we have $13 \mid 7^n - 1$, so $12 \mid n$. Thus,

$$7^{12} - 1 \mid 7^n - 1.$$

Since $43 \mid 7^{12} - 1$, we have $43 \mid 3^m - 1$, and hence $42 \mid m$. Therefore,

$$3^{42} - 1 \mid 3^m - 1.$$

Since $49 \mid 3^{42} - 1$, we obtain $49 \mid 3^m - 1$. It follows from

$$3(3^m - 1) = 7(7^n - 1)$$

that $49 \mid 7(7^n - 1)$, which is impossible because $7 \nmid 7^n - 1$. This completes the case $c = 2$.

Now we can assume that $c \geq 3$, so that $2^c \equiv 0 \pmod 8$. Note that d must be odd, so $d^2 \equiv 1 \pmod 8$. Therefore,

$$3^a 7^b \equiv 1 \pmod 8.$$

Since 3^a cycles $3, 1, 3, 1, \ldots$ and 7^b cycles $7, 1, 7, 1, \ldots$ modulo 8, it must be the case that a and b are both even. Let x and y be such that $a = 2x$ and $b = 2y$. Then we can rearrange and factor the equation as follows:

$$\begin{aligned} 3^{2x} 7^{2y} + 2^c &= d^2, \\ d^2 - 3^{2x} 7^{2y} &= 2^c, \\ (d + 3^x 7^y)(d - 3^x 7^y) &= 2^c. \end{aligned}$$

Then

$$d + 3^x 7^y = 2^z$$

and

$$d - 3^x 7^y = 2^{c-z}$$

for some integer $c/2 < z \le c - 1$. Subtracting the two equations gives

$$2 \cdot 3^x 7^y = 2^z - 2^{c-z} = 2^{c-z}(2^{2z-c} - 1).$$

Since $2^{2z-c} - 1$ is odd, matching the factors of 2 gives $c - z = 1$, or $z = c - 1$. Therefore,

$$3^x 7^y = 2^{2z-c} - 1 = 2^{c-2} - 1.$$

Let $w = c - 2$. To solve the equation

$$3^x 7^y = 2^w - 1,$$

we check the smallest cases:

$$3^0 \cdot 7^0 = 2^1 - 1, \qquad 3^1 \cdot 7^0 = 2^2 - 1, \qquad 3^0 \cdot 7^1 = 2^3 - 1.$$

These yield the triples

$$(x, y, w) = (0, 0, 1), (1, 0, 2), (0, 1, 3),$$

which correspond to

$$(a, b, c) = (0, 0, 3), (2, 0, 4), (0, 2, 5).$$

Then we can use Zsigmondy's theorem. For every $w > 3$ with $w \ne 6$, the number $2^w - 1$ has a primitive prime factor. Such a prime factor divides neither

$$2^2 - 1 = 3$$

nor

$$2^3 - 1 = 7,$$

and therefore $2^w - 1$ cannot equal $3^x 7^y$. The exceptional case is $w = 6$, for which

$$2^6 - 1 = 63 = 3^2 \cdot 7.$$

This gives

$$(x, y, w) = (2, 1, 6),$$

yielding the triple

$$(a, b, c) = (4, 2, 8).$$

Here's a proof without Zsigmondy's theorem. First, check $x = 0$, $y = 0$ to get the triple $(x, y, w) = (0, 0, 1)$, or $(a, b, c) = (0, 0, 3)$.

Whenever w is odd, we have $3 \nmid 2^w - 1$, and whenever w is even, LTE gives

$$\nu_3(2^w - 1) = \nu_3(4^{w/2} - 1) = 1 + \nu_3(w/2) = 1 + \nu_3(w).$$

Whenever w is not a multiple of 3, we have $7 \nmid 2^w - 1$, and whenever w is a multiple of 3, LTE gives

$$\nu_7(2^w - 1) = \nu_7(8^{w/3} - 1) = 1 + \nu_7(w/3) = 1 + \nu_7(w).$$

First, if w is odd, we have $x = 0$, so

$$7^y = 2^w - 1.$$

Since we have already handled $x = y = 0$, we can assume $y \geq 1$. Then w must be a multiple of 3, which gives

$$\nu_7(2^w - 1) = 1 + \nu_7(w),$$

so

$$7^{1+\nu_7(w)} = 2^w - 1.$$

But

$$7^{1+\nu_7(w)} \leq 7 \cdot 7^{\log_7(w)} = 7w,$$

so

$$7w \geq 2^w - 1.$$

For every $w \geq 6$, however, $2^w - 1 > 7w$. Indeed, this holds at $w = 6$, and if it holds for some $w \geq 6$, then

$$2^{w+1} - 1 = 2(2^w - 1) + 1 > 14w + 1 > 7(w + 1).$$

Thus, $w < 6$. Checking the only positive odd multiple of 3 below 6, namely $w = 3$, yields

$$(x, y, w) = (0, 1, 3),$$

or

$$(a, b, c) = (0, 2, 5).$$

Next, suppose that w is even and is not a multiple of 3. Then $y = 0$, so

$$3^x = 2^w - 1.$$

We can assume $x \geq 1$. We have

$$\nu_3(2^w - 1) = 1 + \nu_3(w),$$

so

$$3^{1+\nu_3(w)} = 2^w - 1.$$

But

$$3^{1+\nu_3(w)} \leq 3 \cdot 3^{\log_3(w)} = 3w,$$

so

$$3w \geq 2^w - 1.$$

For every $w \geq 4$, however, $2^w - 1 > 3w$. Indeed, this holds at $w = 4$, and if it holds for some $w \geq 4$, then

$$2^{w+1} - 1 = 2(2^w - 1) + 1 > 6w + 1 > 3(w + 1).$$

Thus, $w < 4$. Checking the only remaining positive even possibility, $w = 2$, yields

$$(x, y, w) = (1, 0, 2),$$

or

$$(a, b, c) = (2, 0, 4).$$

Finally, if w is a multiple of 6, we have both

$$\nu_3(2^w - 1) = 1 + \nu_3(w)$$

and

$$\nu_7(2^w - 1) = 1 + \nu_7(w),$$

so

$$3^{1+\nu_3(w)}7^{1+\nu_7(w)} = 2^w - 1.$$

Then

$$3^{1+\nu_3(w)}7^{1+\nu_7(w)} \leq (3w)(7w) = 21w^2,$$

so

$$21w^2 \geq 2^w - 1.$$

For every $w \geq 12$, however, $2^w - 1 > 21w^2$. This holds at $w = 12$, since

$$2^{12} - 1 = 4095 > 3024 = 21 \cdot 12^2.$$

Moreover, if it holds for some $w \geq 12$, then

$$2^{w+1} - 1 = 2(2^w - 1) + 1 > 42w^2 + 1 > 21(w + 1)^2.$$

Thus, $w < 12$. Checking the only positive multiple of 6 below 12, namely $w = 6$, yields

$$(x, y, w) = (2, 1, 6),$$

or

$$(a, b, c) = (4, 2, 8).$$

This completes the case $c \geq 3$.

In all, the possible triples (a, b, c) are

$$(1, 0, 0),\ (2, 1, 0),\ (0, 1, 1),\ (1, 1, 2),\ (0, 0, 3),\ (2, 0, 4),\ (0, 2, 5),\ (4, 2, 8).$$

Therefore, the complete set of solutions (p, q, r) is

$$(0, 0, 5),\ (3, 0, 0),\ (3, 1, 0),\ (1, 1, 1),\ (2, 1, 2),\ (2, 0, 3),\ (4, 0, 4),\ (0, 2, 5),\ (4, 2, 8).$$

These give the respective values

$$d = 17,\ 6,\ 24,\ 9,\ 15,\ 9,\ 15,\ 27,\ 195,$$

so direct substitution confirms that every listed triple is a solution.

Plugging these triples into the expression $100p + 10q + r$ and summing gives

$$5 + 300 + 310 + 111 + 212 + 203 + 404 + 25 + 428 = \boxed{1998}.$$