1. Compute
$$5^2 + 8^2 - 2 \cdot 5 \cdot 8 + 58.$$

   Answer: 67

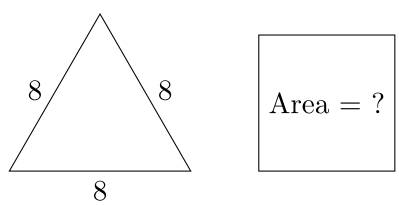
   Solution 1: Compute the expression directly:
$$\begin{aligned} 5^2 + 8^2 - 2 \cdot 5 \cdot 8 + 58 &= 25 + 64 - 80 + 58 \\ &= 89 - 80 + 58 \\ &= 9 + 58 \\ &= \boxed{67}. \end{aligned}$$

   Solution 2: Alternatively, we can use the fact that $(5-8)^2 = 5^2 - 2 \cdot 5 \cdot 8 + 8^2$:
$$\begin{aligned} 5^2 + 8^2 - 2 \cdot 5 \cdot 8 + 58 &= (5^2 - 2 \cdot 5 \cdot 8 + 8^2) + 58 \\ &= (5-8)^2 + 58 \\ &= 3^2 + 58 \\ &= \boxed{67}. \end{aligned}$$

2. An equilateral triangle and a square have the same perimeter. Given that the sides of the triangle each have length 8, compute the area of the square.

   Answer: 36

   Solution:



   The triangle has side length 8, so its perimeter is $8 + 8 + 8 = 24$. Then the square must also have perimeter 24, so each of its sides has length $\frac{24}{4} = 6$. The area of the square is $6^2 = \boxed{36}$.

3. Find the smallest two-digit positive integer whose digits sum to a multiple of 6 and whose ones digit is twice its tens digit.

   Answer: 24

   Solution: A good approach is to count up through positive integers whose ones digit is twice the tens digit.

   - If the tens digit is 1, then the ones digit is 2. The number 12 doesn't satisfy the first condition since $1 + 2 = 3$ is not a multiple of 6.
   - If the tens digit is 2, then the ones digit is 4. The number $\boxed{24}$ satisfies the first condition since $2 + 4 = 6$ is a multiple of 6.

   At this point we know we're done since if we continued increasing the tens digit, we would certainly end up with a number greater than 24.

4. Ylann has four shirts (red, orange, yellow, and green) and two pairs of pants (black and white). An outfit consists of one shirt and one pair of pants. If Ylann refuses to wear a red shirt with white pants, and he also refuses to wear a yellow shirt with black pants, how many acceptable outfits can he make?

Answer: 6

Solution: Let's temporarily ignore the fact that there are some combinations Ylann refuses to wear. He has four ways to choose a shirt and two ways to choose a pair of pants, so there are $4 \times 2 = 8$ total outfits he can make. Out of these possibilities, two are "forbidden" by Ylann's fashion preferences, so in total there are $8 - 2 = \boxed{6}$ acceptable outfits Ylann can make.

5. Oliver writes the number 1 on the board and then makes a series of moves. In each move, Oliver either multiplies the number on the board by 3 or increases it by 20. What is the minimum number of moves Oliver needs to change the number on the board to 67?

Answer: 5

Solution: Work backward from 67. Since 67 and 47 are not divisible by 3, the last two moves must have been adding 20:

$$67 \longleftarrow 47 \longleftarrow 27.$$

27 may have come after 9 or 7. Note 7 can't be reached from 1, so the remaining steps are forced as well:

$$27 \longleftarrow 9 \longleftarrow 3 \longleftarrow 1.$$

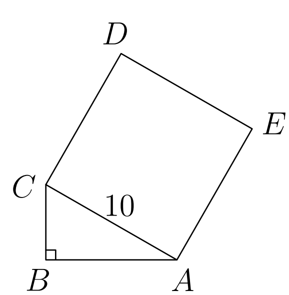
The minimum number of moves is $\boxed{5}$.

6. Jonathan and Carlyana race around a circular track with circumference 40 meters. They run clockwise at constant speeds: Jonathan runs at 5 meters per second, and Carlyana runs at 10 meters per second. They clap whenever they are at the same position on the track at the same time. If the first clap occurs at the start of the race, how many seconds after the start of the race does the second clap occur?

Answer: 8

Solution: Carlyana meets Jonathan again when she has run one full lap more than him. That is, at the point when Carlyana laps Jonathan, Carlyana has run 40 meters more than Jonathan. Each second, Carlyana runs 5 more meters than Jonathan runs, so the second clap occurs after $\frac{40}{5} = \boxed{8}$ seconds.

7. Triangle $\triangle ABC$ has a right angle at B and all integer side lengths. Square $ACDE$ is constructed outside $\triangle ABC$. Given $AC = 10$, what is the perimeter of pentagon $ABCDE$?



Answer: 44

Solution: Notice that 6 - 8 - 10 is the only possible combination of lengths of a right triangle with integer side lengths and hypotenuse 10. Using the fact that the side length of the square $ACDE$ is 10, the perimeter of the pentagon is $6 + 8 + 10 + 10 + 10 = \boxed{44}$.

Here's an argument that 6 - 8 - 10 is the only case to consider:

Since $\triangle ABC$ is a right triangle, the Pythagorean Theorem gives $AB^2 + BC^2 = 10^2 = 100$. Assume without loss of generality that $AB \leq BC$. Let's do casework on the possible values of AB:

- $AB = 1$, then $BC^2 = 100 - 1^2 = 99$, then BC is not an integer!
- $AB = 2$, then $BC^2 = 100 - 2^2 = 96$, then BC is not an integer!
- $AB = 3$, then $BC^2 = 100 - 3^2 = 91$, then BC is not an integer!
- $AB = 4$, then $BC^2 = 100 - 4^2 = 84$, then BC is not an integer!
- $AB = 5$, then $BC^2 = 100 - 5^2 = 75$, then BC is not an integer!
- $AB = 6$, then $BC^2 = 100 - 6^2 = 64$, then $BC = 8$ is an integer!
- $AB = 7$, then $BC^2 = 100 - 7^2 = 51$, then BC is not an integer!
- $AB \geq 8$, then $BC^2 = 100 - AB^2 < 64$, then $BC < AB$!

8. Jordan has two white socks and two black socks in their sock drawer. If they pull out two socks uniformly at random without replacement, what is the probability that they are the same color?

   Answer: $\frac{1}{3}$

   Solution: After the first sock is pulled out, there are three other socks in the drawer. Only one of the socks left in the drawer is the same color as the first sock that was taken. Thus, the probability that the second sock matches the first is $\boxed{\frac{1}{3}}$.

9. A hat initially contains five slips labeled with the numbers 1, 2, 3, 4, and 5 (one slip per number). Every minute, Jessica adds a new slip labeled with the number 20 to the hat. How many slips are in the hat when the average of the numbers in the hat is 5 times the initial average?

   Answer: 17

   Solution: The initial average is

$$\frac{1+2+3+4+5}{5} = \frac{15}{5} = 3.$$

Let x be the number of minutes until the average of the numbers is 5 times the initial average. We can write the following equation and solve for x:

$$\begin{aligned} 5 \cdot 3 &= \frac{15+20x}{5+x} \\ 15 &= \frac{15+20x}{5+x} \\ 15 \cdot (5+x) &= 15+20x \\ 75+15x &= 15+20x \\ 60 &= 5x \\ 12 &= x. \end{aligned}$$

Therefore, it takes 12 minutes until the average is five times the initial average. The hat started with 5 slips and 12 more are added, so at the end there are $\boxed{17}$ slips in the hat.

10. Let A, B, and C be digits such that $\underline{A}\,\underline{B}\,\underline{C} \times \underline{A}\,\underline{C} = 2025$. Find the three-digit number $\underline{A}\,\underline{B}\,\underline{C}$.

    Answer: 135

    Solution: Note that the units digit of the product $\underline{A}\,\underline{B}\,\underline{C} \times \underline{A}\,\underline{C}$ is 5. This implies the units digit of at least one of $\underline{A}\,\underline{B}\,\underline{C}$ and $\underline{A}\,\underline{C}$ is 5, so $C = 5$.

    If $A \geq 2$, the product would be at least $200 \times 20 = 4000 > 2025$. Similarly, if $A < 1$, then $A = 0$ and the product is too small. Thus, $A = 1$.

    Finally, $\underline{1}\,\underline{B}\,\underline{5} \times \underline{1}\,\underline{5} = 2025 \implies \underline{A}\,\underline{B}\,\underline{C} = \dfrac{2025}{15} = \boxed{135}$.

11. Find the real number x satisfying $\sqrt{x+3} - \sqrt{x-1} = 1$.

    Answer: $\mathbf{\frac{13}{4}}$, 3.25

    Solution: Manipulating the equation gives

$$\begin{aligned} \sqrt{x+3} &= 1 + \sqrt{x-1} \\ x+3 &= 1 + 2\sqrt{x-1} + x - 1 \\ 3 &= 2\sqrt{x-1} \\ \frac{3}{2} &= \sqrt{x-1} \\ \frac{9}{4} &= x - 1 \\ \frac{9}{4} + 1 &= \boxed{\frac{13}{4}} = x. \end{aligned}$$

12. Find the positive integer n such that $n + 10$ is both a multiple of n and a factor of $n + 100$.

    Answer: 5

    Solution: Note that if n and x are integers such that $n + x$ is a multiple of n, then x is a multiple of n. In particular, if $n + x = ny$ for some integer y, then $x = ny - n = n(y - 1)$, which is a multiple of n.

    Thus, if $n + 10$ is a multiple of n, then 10 is a multiple of n, so $n = 1, 2, 5$, or 10.

    Similarly, if $n+10$ is a factor of $n+100 = (n+10)+90$, then 90 is a multiple of $(n+10)$. Trying $n = 1, 2, 5, 10$ gives $n + 10 = 11, 12, 15, 20$, of which only 15 divides 90. Thus, $n = \boxed{5}$.

13. An arithmetic sequence is a sequence of numbers where the difference between any two consecutive terms is the same. Harsh writes an arithmetic sequence where the ratio of the second term to the first term is 4, and the sum of the third and fourth terms is 34. Compute the second term of Harsh's sequence.

    Answer: 8

    Solution: Let the first term be a, and let the common difference be d. Then the second, third, and fourth terms are $a + d$, $a + 2d$, and $a + 3d$, respectively. We are given

$$\frac{a+d}{a} = 4 \quad \text{and} \quad (a + 2d) + (a + 3d) = 34.$$

We can solve for d in terms of a from the first equation:

$$\begin{aligned}\frac{a+d}{a} &= 4\\ a+d &= 4a\\ d &= 3a.\end{aligned}$$

Plugging this into the second equation gives

$$\begin{aligned}(a+2d)+(a+3d) &= 34\\ 2a+5d &= 34\\ 2a+5\cdot(3a) &= 34\\ 17a &= 34\\ a &= 2.\end{aligned}$$

Using $d = 3a$ and $a = 2$ gives $d = 6$. Thus, the second term of the sequence is $a + d = \boxed{8}$.

14. A square of area A is divided using 2 lines into 4 rectangles (which may be squares) with all integer side lengths. Two of the rectangles have areas of 4 and 9, respectively. Find the sum of both distinct possible values of A.

Answer: 194

Solution: If the rectangles of areas 4 and 9 share a side, that side has length equal to an integer that divides both 4 and 9, so that side has length 1. The rectangles are therefore 1×4 and 1×9, and the square has side length $4 + 9 = 13$ with area 169.

Otherwise, the two rectangles are diagonally opposite. Write their dimensions as $x\times y$ and $u\times v$, where $xy = 4$ and $uv = 9$. Since opposite rectangles fill the same square,

$$x+u = y+v.$$

Checking the integer factor pairs of 4 and 9, the only possibility is

$$(x,y) = (2,2), \qquad (u,v) = (3,3).$$

This gives a square of side length 5 and area 25. Thus the sum of the two possible areas is

$$169+25 = \boxed{194}.$$

15. Ariel places 4 green stickers, 4 orange stickers, and 4 purple stickers on the faces of two fair 6-sided dice such that each face has exactly one sticker, and each die has at least one sticker of every color. If these dice are rolled, the probability that two green faces are rolled is $\frac{1}{12}$, and the probability that two orange faces are rolled is $\frac{1}{12}$. What is the probability that two purple faces are rolled?

Answer: $\mathbf{\frac{1}{9}}$

Solution: For each color, the sticker split between the dice is either 1 to 3 or 2 to 2. If the sticker split is 1 to 3, the probability that the results of both dice are that color is $\dfrac{1}{6}\cdot\dfrac{3}{6} = \dfrac{3}{36} = \dfrac{1}{12}$. If the sticker split is 2 to 2, the probability that the results of both dice are that color is $\dfrac{2}{6}\cdot\dfrac{2}{6} = \dfrac{4}{36} = \dfrac{1}{9}$.

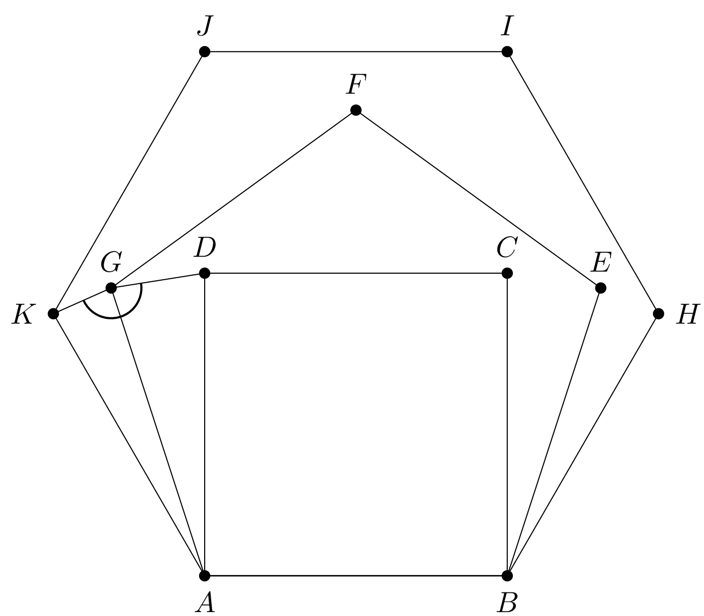
Since the probabilities for green and for orange are both $\frac{1}{12}$, the sticker splits for green and orange are both 1 to 3. So one die has 1 green sticker and 3 orange stickers, and the other has 3 green stickers and 1 orange sticker; otherwise one die has 3 green stickers and 3 orange stickers, which leaves no room for a purple sticker.

This means that the sticker split for purple must be 2 to 2. So the probability that the results of both dice are purple is $\boxed{\frac{1}{9}}$.

16. Square $ABCD$, regular pentagon $ABEFG$, and regular hexagon $ABHIJK$ all lie in the same plane so that the square lies inside the pentagon, and the pentagon lies inside the hexagon. Compute the degree measure of $\angle DGK$ which is less than $180°$.

Answer: 165°

Solution:



Draw line segments $\overline{DG}$ and $\overline{GK}$ to create triangles $\triangle ADG$ and $\triangle AGK$. Since the polygons are regular and share a side, their side lengths are all equal. In particular, $\overline{AD} = \overline{AG} = \overline{AK}$, so $\triangle ADG$ and $\triangle AGK$ are isosceles.

The fact that the interior angles of a regular n-gon measure $\frac{n-2}{n} \cdot 180°$ implies $m\angle DAB = 90°$, $m\angle GAB = 108°$, and $m\angle KAB = 120°$. From this, we calculate that $m\angle GAD = m\angle GAB - m\angle DAB = 18°$ and $m\angle KAG = m\angle KAB - m\angle GAB = 12°$.

Because $\triangle ADG$ and $\triangle AGK$ are isosceles, $m\angle ADG = m\angle AGD$ and $m\angle AKG = m\angle AGK$. Therefore, $m\angle AGD = \frac{180-18}{2}° = 81°$ and $m\angle AGK = \frac{180-12}{2}° = 84°$.

Finally, the degree measure of $\angle DGK$ is $m\angle AGD + m\angle AGK = 81° + 84° = \boxed{165°}$.

17. In how many ways can ten identical white blocks be placed into a red box, a yellow box, a green box, and a blue box such that no box is empty, and at least two boxes contain the same number of blocks?

Answer: 60

Solution 1: List out the different arrangements up to permutation, taking care not to double-count:

$$\{1,1,1,7\},\{1,1,2,6\},\{1,1,3,5\},\{1,1,4,4\},\{2,2,1,5\},\{2,2,2,4\},\{3,3,2,2\},\{3,3,3,1\}.$$

There are three cases where three boxes have the same number of blocks in them, which can happen $\frac{4!}{3!1!} = 4$ ways each. There are two cases where there are two pairs of boxes with an equal number of blocks between them, which can happen $\frac{4!}{2!2!} = 6$ different ways each. Lastly, there are three cases where two boxes have the same number of blocks, and the remaining two boxes have distinct numbers of blocks. This can happen $\frac{4!}{2!1!1!} = 12$ different ways for each of the three cases. Thus, there are $(3 \cdot 4) + (2 \cdot 6) + (3 \cdot 12) = \boxed{60}$ arrangements.

Solution 2: If we ignore the condition that two boxes have the same number of blocks, we can use stars and bars to show that there are $\binom{10-1}{4-1} = \binom{9}{3} = 84$ distinct ways to distribute ten identical blocks among four different boxes. The only case where the number of blocks in each of the four boxes is distinct is $\{1,2,3,4\}$, and there are $4! = 24$ ways to arrange this. Thus, the number of arrangements that satisfy the given conditions is $84 - 24 = \boxed{60}$.

18. Benji writes down the decimal representation of a rational number greater than 10. He observes that if he replaces the tens digit with 2, the number decreases by 70%. If he instead replaces the tens digit with 8, the number increases by 14%. By what percent would the number decrease if he replaced the tens digit with 5?

Answer: 28

Solution 1: Let the given rational number be n, and let t be its tens digit. The result of replacing the tens digit with 2 is $n - 10(t-2) = n - 10t + 20$. The result of replacing the tens digit with 5 is $n - 10(t-5) = n - 10t + 50$. The result of replacing the tens digit with 8 is $n - 10(t-8) = n - 10t + 80$.

We get the system of equations:

$$n - 10t + 20 = 0.3n$$
$$n - 10t + 80 = 1.14n$$

We don't even need to solve for n or t: we just need to find $n - 10t + 50$ as a multiple of n. Averaging the equations (adding them together and dividing by 2) gives:

$$n - 10t + 50 = \frac{0.3n + 1.14n}{2} = 0.72n.$$

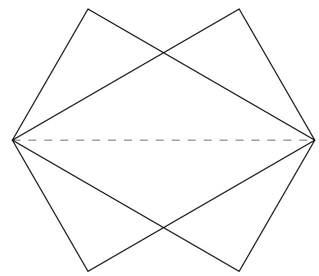
Therefore, replacing the tens digit with 5 leads to a $100 - 72 = \boxed{28}$ percent decrease.

Solution 2: By linearity, the result of replacing the tens digit by 5 is the average of the results obtained by replacing the tens digit by 2 or 8. Thus, the percent change due to replacing the tens digit by 5 is the average of the percent change in the other two cases. The percent change is then $\frac{-70\%+14\%}{2} = -28\%$ so the answer is $\boxed{28}$.

19. A rectangle is reflected over one of its diagonals, resulting in a second rectangle. Given that the region inside both of these rectangles is a rhombus with perimeter 52 and area 120, what is the perimeter of the original rectangle?

Answer: $\mathbf{\frac{816}{13}}$

Solution:



Let $ABCD$ be the original rectangle, with $AB < BC$, and let rectangle $AECF$ be the reflection of $ABCD$ about the diagonal $\overline{AC}$. Let $\overline{AD}$ and $\overline{CE}$ intersect at point G.

Since the perimeter of the rhombus is 52, each side of the rhombus has length $\frac{52}{4} = 13$. CD is the height of the rhombus, and the length of the base is 13, so $CD \cdot 13 = 120 \implies CD = \dfrac{120}{13}$.

Then, triangle CDG has a right angle at D, with side lengths $CD = \dfrac{120}{13}$, GD, and $CG = 13$. By the Pythagorean theorem, we have:

$$\begin{aligned} GD^2 &= CG^2 - CD^2 \\ &= 13^2 - \left(\frac{120}{13}\right)^2 = \frac{169^2}{13^2} - \frac{120^2}{13^2} \\ &= \frac{169^2 - 120^2}{13^2} = \frac{(169+120)(169-120)}{13^2} \\ &= \frac{289 \cdot 49}{13^2} = \frac{17^2 \cdot 7^2}{13^2} = \frac{119^2}{13^2}. \end{aligned}$$

Therefore, $GD = \dfrac{119}{13}$, so

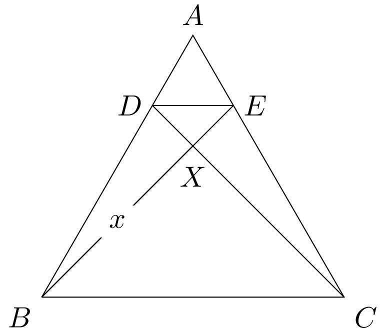
$$AD = AG + GD = 13 + \frac{119}{13} = \frac{288}{13}.$$

The perimeter of rectangle $ABCD$ is $2 \cdot \left(\dfrac{288}{13} + \dfrac{120}{13}\right) = \boxed{\dfrac{816}{13}}$.

20. Let triangle $\triangle ABC$ be equilateral. Points D and E lie on sides $\overline{AB}$ and $\overline{AC}$, respectively, such that $\overline{DE} \parallel \overline{BC}$ and $\overline{CD} \perp \overline{BE}$. Compute $\frac{AD}{AB}$.

Answer: $\mathbf{2 - \sqrt{3}}$ or $\mathbf{\frac{1}{2+\sqrt{3}}}$

Solution:



Since $\overline{DE} \parallel \overline{BC}$, triangle $\triangle ADE$ is similar to equilateral triangle $\triangle ABC$. For convenience, let $AD = DE = AE = \sqrt{2}$ (we can scale the diagram however we like since no lengths are given). Let X be the intersection of CD and BE. By symmetry, $\triangle DEX$ and $\triangle BCX$ are isosceles right triangles, so $DX = 1$. Let $BX = x$. Then $BC = AB = x\sqrt{2}$.

The Pythagorean Theorem on right triangle $\triangle BDX$ gives $BD = \sqrt{x^2+1}$. Furthermore,

$$AB = AD + DB \implies x\sqrt{2} = \sqrt{2} + \sqrt{x^2+1}$$

which has candidate solutions $x = 2 \pm \sqrt{3}$. If $x = 2 - \sqrt{3}$, then $BD = (1-\sqrt{3})\sqrt{2} < 0$, which is impossible, so $x = 2+\sqrt{3}$.

Finally,

$$\frac{AD}{AB} = \frac{\sqrt{2}}{\sqrt{2}(2+\sqrt{3})} = \boxed{\frac{1}{2+\sqrt{3}}} = \boxed{2-\sqrt{3}}.$$

21. A *substring* of a positive integer n is a nonnegative integer whose digits appear consecutively in the digits of n. For example, 10, $00 = 0$, and $02 = 2$ are two-digit substrings of 1002, but 12 and $01 = 1$ are not. An integer n is chosen uniformly at random from the integers between 1000 and 9999 inclusive. Compute the probability that all two-digit substrings of n are divisible by three.

Answer: $\mathbf{\frac{59}{1500}}$

Solution: There are $9999 - 1000 + 1 = 9000$ integers that could be chosen, each with equal probability. We proceed by counting the number of these integers n that have all two-digit substrings divisible by 3. Write n as $\overline{d_3d_2d_1d_0}$, where we allow digits other than d_3 to be zero. For $k = 0, 1,$ or 2, the number $\overline{d_{k+1}d_k}$ must be divisible by three, which is equivalent to requiring $d_k + d_{k+1}$ to be divisible by 3. We now have the following cases.

- Suppose d_0 is divisible by 3. Then all digits are divisible by 3, so each has four options among $\{0, 3, 6, 9\}$, except d_3 must be nonzero. Thus, there are $3 \cdot 4^3 = 3 \cdot 64 = 192$ positive integers here.
- Suppose d_0 is not divisible by 3. Then d_0 and d_2 leave the same remainder when divided by 3, and likewise for d_1 and d_3; additionally, $d_0 + d_1$ is divisible by 3. There are 6 options for d_0, and then each remaining digit has 3 options, so there are $6 \cdot 3^3 = 6 \cdot 27 = 162$ positive integers here.

There are $192 + 162 = 354$ values of n which meet our criteria. The probability of choosing one of these values is $\frac{354}{9000} = \boxed{\frac{59}{1500}}$.

22. Over all triples of positive integers (a, b, c) satisfying the equation

$$\gcd(\operatorname{lcm}(a,b), \operatorname{lcm}(a,c), \operatorname{lcm}(b,c)) = 2025,$$

compute the smallest possible value of $a + b + c$.

Answer: 2131

Solution: Note that $2025 = 3^4 \cdot 5^2$ divides $\operatorname{lcm}(a,b)$, $\operatorname{lcm}(a,c)$, and $\operatorname{lcm}(b,c)$. Thus in each pair (a,b), (a,c) and (b,c), at least one member of the pair must be divisible by 3^4 and at least one member of the pair must be divisible by 5^2.

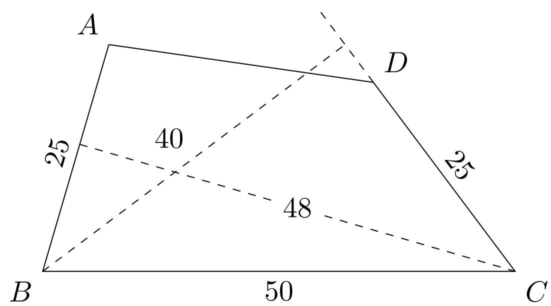
By the pigeonhole principle, at least one of a, b, and c is divisible by both 3^4 and 5^2. Without loss of generality, let c be divisible by 3^4 and 5^2, so $c \geq 3^4 \cdot 5^2 = 2025$.

At least one of a and b must be divisible by 3^4, and at least one must be divisible by 5^2. There are two cases: at least one of a, b is divisible by both 3^4 and 5^2, or one is divisible by 3^4 but not 5^2, and the other is divisible by 5^2 but not 3^4.

The second case yields a and b which have a smaller sum: 81 and 25 (the other case gives a minimum value of 2025 for one of a or b).

Thus, the smallest possible value of $a + b + c$ is $81 + 25 + 2025 = \boxed{2131}$.

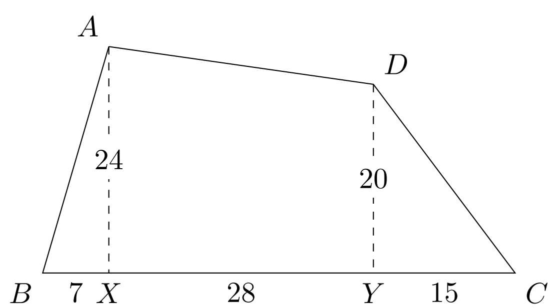
23. Quadrilateral $ABCD$ has $AB = CD = 25$, $BC = 50$, and acute angles at vertices B and C. The distance from C to line $\overleftrightarrow{AB}$ is 48, and the distance from B to line $\overleftrightarrow{CD}$ is 40. Compute the area of quadrilateral $ABCD$.



Answer: 850

Solution: First, we use the areas of triangles $\triangle ABC$ and $\triangle BCD$ to find the heights relative to the base $\overline{BC}$. Since $BC = 2AB$, the height from A to $\overline{BC}$ is half of the height from C to $\overleftrightarrow{AB}$, which is 24. Similarly, since $BC = 2CD$, the height from D to $\overline{BC}$ is half of the height from B to $\overleftrightarrow{CD}$, which is 20.

Let X and Y be the feet of the altitudes from A and D, respectively, to $\overline{BC}$. Then $AX = 24$ and $DY = 20$. Because the angles at B and C are acute, X and Y both lie on segment $\overline{BC}$. By the Pythagorean theorem, $BX = \sqrt{25^2 - 24^2} = 7$ and $CY = \sqrt{25^2 - 20^2} = 15$, so $XY = BC - BX - CY = 50 - 7 - 15 = 28$.



Then quadrilateral $ABCD$ can be partitioned into triangle $\triangle AXB$, triangle $\triangle DYC$, and trapezoid $AXYD$. The area of $\triangle AXB$ is $\frac{1}{2} \cdot 7 \cdot 24 = 84$, the area of $\triangle DYC$ is $\frac{1}{2} \cdot 15 \cdot 20 = 150$, and the area of trapezoid $AXYD$ is $\frac{24+20}{2} \cdot 28 = 616$. Thus, the area of quadrilateral $ABCD$ is $84 + 150 + 616 = \boxed{850}$.

24. Compute the sum of all distinct real numbers x satisfying $|x| \leq \frac{1}{2}$ and

$$\sin\left(\log_{10}\left(\frac{1}{x} + \frac{1}{x^2}\right)\right) = 0.$$

Note that the argument of the sine function is assumed to be in radians.

Answer: $\frac{\mathbf{1}}{\mathbf{10^{\pi}-1}}$

Solution: Since $\sin\theta = 0$ exactly when $\theta = k\pi$ for an integer k, we have

$$\log_{10}\left(\frac{1}{x}+\frac{1}{x^2}\right) = k\pi \quad \text{for some integer } k.$$

Exponentiating gives

$$\frac{1}{x}+\frac{1}{x^2} = 10^{k\pi}.$$

The left side is undefined at $x = 0$. On $[-1/2, 0)$ it is increasing from 2 to ∞, while on $(0, 1/2]$ it is decreasing from ∞ to 6. For integers $k \geq 1$ there are two different solutions, and for integers $k \leq 0$ there are no solutions. Rewrite the equation in standard quadratic form:

$$\frac{1}{x}+\frac{1}{x^2} = 10^{k\pi} \quad \Longrightarrow \quad 10^{k\pi}x^2 - x - 1 = 0.$$

Since the discriminant $\Delta = 1 + 4 \cdot 10^{k\pi}$ of this quadratic is always positive, there are two real roots for each k. By Vieta's formulas, the sum of those two roots is

$$\frac{-(-1)}{10^{k\pi}} = 10^{-k\pi}.$$

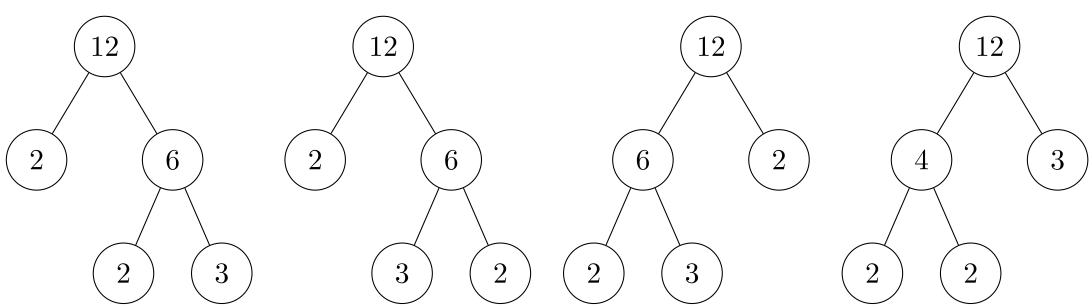
Another way to see this is by observing that the two roots are given by $\frac{1\pm\sqrt{\Delta}}{2\cdot 10^{k\pi}}$, so the sum of the two roots is $\frac{2}{2\cdot 10^{k\pi}} = \frac{1}{10^{k\pi}}$.

Finally, summing over all integers $k \geq 1$ gives the infinite geometric series

$$\sum_{k=1}^{\infty} 10^{-k\pi} = \frac{10^{-\pi}}{1-10^{-\pi}} = \boxed{\frac{1}{10^{\pi}-1}}.$$

25. A *factor tree* for a composite number n is a binary tree with root n and two children, a and b, such that a and b are proper divisors of n with $ab = n$, and whose subtrees rooted at a and b are themselves factor trees for a and b, respectively. A factor tree for a prime number is the single node containing the number itself.

Four distinct factor trees for 12 are shown below.



How many distinct factor trees are there for 120?

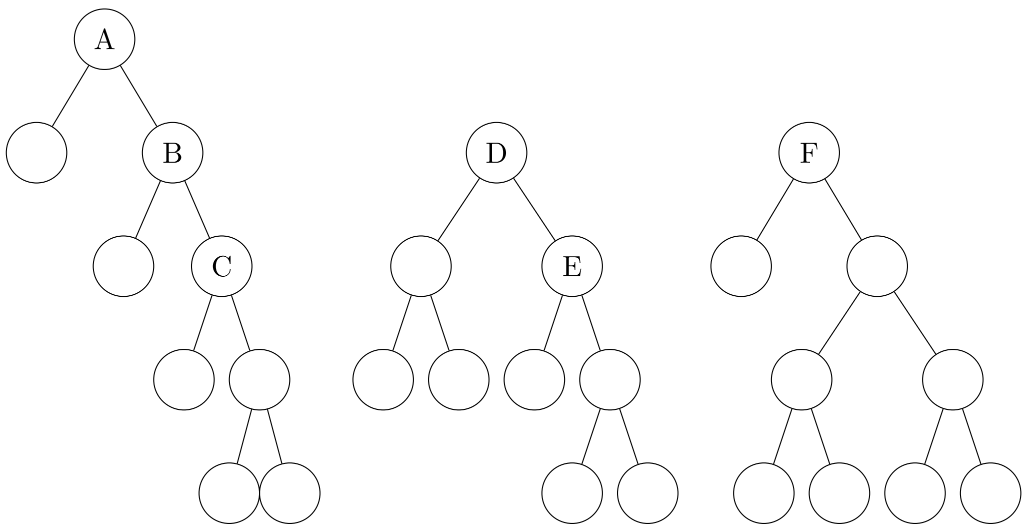
Answer: 280

Solution 1: For each integer n, let $F(n)$ be the number of distinct factor trees for n. When constructing a factor tree for a composite number, we need to choose a left branch and a right branch. Call the divisor that starts the left branch d. Then the divisor that starts the right branch is n/d. For this choice of d, there are $F(d)$ possible left branches; similarly, there are $F(n/d)$ right branches. Thus, each proper divisor d of n contributes $F(d) \cdot F(n/d)$ trees to the total. We can recursively evaluate any value of $F(n) = \sum_{d|n, 1<d<n} F(d) \cdot F(n/d)$.

We can work our way up from the prime numbers to build the following table; primes are good places to start because $F(p) = 1$ for every prime p. We find that

n	2	3	4	5	6	8	10	12	15	20	24	30	40	60	120
F(n)	1	1	1	1	2	2	2	6	2	6	20	12	20	60	$\boxed{280}$

Solution 2: $120 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$, so in every factor tree for 120, the numbers $2, 2, 2, 3, 5$ occur with multiplicity at the tips of the branches, which are usually called *leaves*. A full rooted binary tree with 5 leaves can have one of three fundamental structures:



The structure with root A can take 2^3 shapes by swapping the left and right subtrees at nodes A, B, and C. The structure with root D can take 2^2 shapes by swapping the left and right subtrees at nodes D and E. The structure with root F can take 2 shapes by swapping the left and right subtrees at node F.

Thus, there are $8 + 4 + 2 = 14$ different shapes for the tree.

There are $\frac{5!}{3!} = 20$ arrangements for $2, 2, 2, 3, 5$ at five distinct locations, so there are $14 \cdot 20 = \boxed{280}$ distinct factor trees.