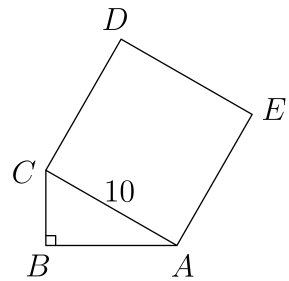
1. Triangle $\triangle ABC$ has a right angle at B and all integer side lengths. Square $ACDE$ is constructed outside $\triangle ABC$. Given $AC = 10$, what is the perimeter of pentagon $ABCDE$?



Answer: 44

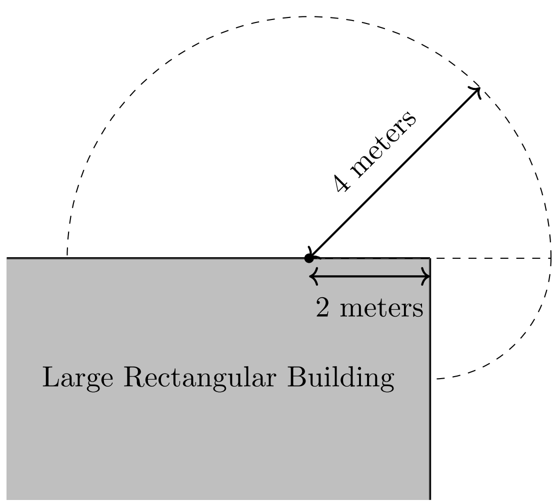
Solution: Notice that 6 - 8 - 10 is the only possible combination of lengths of a right triangle with integer side lengths and hypotenuse 10. Using the fact that the side length of the square $ACDE$ is 10, the perimeter of the pentagon is $6 + 8 + 10 + 10 + 10 = \boxed{44}$.

Here's an argument that 6 - 8 - 10 is the only case to consider:

Since $\triangle ABC$ is a right triangle, the Pythagorean Theorem gives $AB^2 + BC^2 = 10^2 = 100$. Assume without loss of generality that $AB \leq BC$. Let's do casework on the possible values of AB:

- $AB = 1$, then $BC^2 = 100 - 1^2 = 99$, then BC is not an integer!
- $AB = 2$, then $BC^2 = 100 - 2^2 = 96$, then BC is not an integer!
- $AB = 3$, then $BC^2 = 100 - 3^2 = 91$, then BC is not an integer!
- $AB = 4$, then $BC^2 = 100 - 4^2 = 84$, then BC is not an integer!
- $AB = 5$, then $BC^2 = 100 - 5^2 = 75$, then BC is not an integer!
- $AB = 6$, then $BC^2 = 100 - 6^2 = 64$, then $BC = 8$ is an integer!
- $AB = 7$, then $BC^2 = 100 - 7^2 = 51$, then BC is not an integer!
- $AB \geq 8$, then $BC^2 = 100 - AB^2 < 64$, then $BC < AB$!

2. Stephanie uses a rope of length 4 meters to tie a sheep to a point that is 2 meters away from the corner of a large rectangular building. Given the sheep cannot enter the building, what is the area, in square meters, that the sheep can graze?



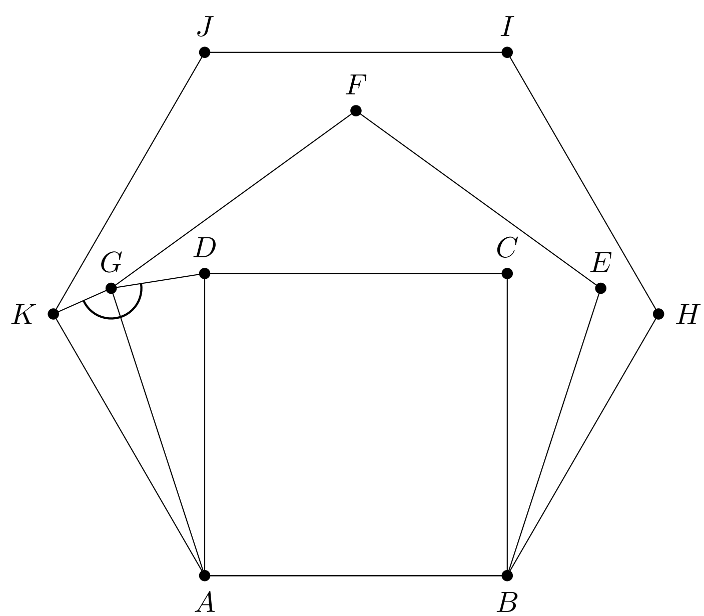
Answer: 9π

Solution: The grazable area consists of a semicircle with radius 4 and a quarter circle with radius $4 - 2 = 2$. The semicircle corresponds to the area the sheep can graze on the side of the building where the rope is tethered, and the quarter circle corresponds to the area the sheep can graze on the other side of the building around the corner. The rope is straight when the sheep is grazing above, and the rope is bent at the corner when the sheep is grazing around the corner, so the effective remaining length of the rope is $4 - 2 = 2$ when the sheep is grazing around the corner. Therefore, the total area is $\frac{\pi \cdot 4^2}{2} + \frac{\pi \cdot 2^2}{4} = \boxed{9\pi}$.

3. Square $ABCD$, regular pentagon $ABEFG$, and regular hexagon $ABHIJK$ all lie in the same plane so that the square lies inside the pentagon, and the pentagon lies inside the hexagon. Compute the degree measure of $\angle DGK$ which is less than $180°$.

Answer: $165°$

Solution:



Draw line segments $\overline{DG}$ and $\overline{GK}$ to create triangles $\triangle ADG$ and $\triangle AGK$. Since the polygons are regular and share a side, their side lengths are all equal. In particular, $\overline{AD} = \overline{AG} = \overline{AK}$, so $\triangle ADG$ and $\triangle AGK$ are isosceles.

The fact that the interior angles of a regular n-gon measure $\frac{n-2}{n} \cdot 180°$ implies $m\angle DAB = 90°$, $m\angle GAB = 108°$, and $m\angle KAB = 120°$. From this, we calculate that $m\angle GAD = m\angle GAB - m\angle DAB = 18°$ and $m\angle KAG = m\angle KAB - m\angle GAB = 12°$.

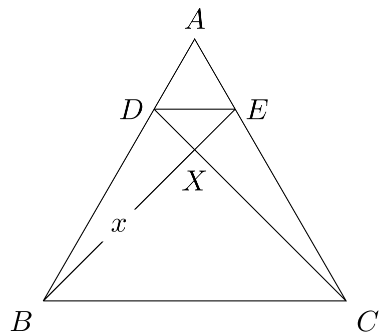
Because $\triangle ADG$ and $\triangle AGK$ are isosceles, $m\angle ADG = m\angle AGD$ and $m\angle AKG = m\angle AGK$. Therefore, $m\angle AGD = \frac{180-18}{2}^{\circ} = 81°$ and $m\angle AGK = \frac{180-12}{2}^{\circ} = 84°$.

Finally, the degree measure of $\angle DGK$ is $m\angle AGD + m\angle AGK = 81° + 84° = \boxed{165°}$.

4. Let triangle $\triangle ABC$ be equilateral. Points D and E lie on sides $\overline{AB}$ and $\overline{AC}$, respectively, such that $\overline{DE} \parallel \overline{BC}$ and $\overline{CD} \perp \overline{BE}$. Compute $\frac{AD}{AB}$.

Answer: $2 - \sqrt{3}$ or $\frac{1}{2+\sqrt{3}}$

Solution:



Since $\overline{DE} \parallel \overline{BC}$, triangle $\triangle ADE$ is similar to equilateral triangle $\triangle ABC$. For convenience, let $AD = DE = AE = \sqrt{2}$ (we can scale the diagram however we like since no lengths are given). Let X be the intersection of CD and BE. By symmetry, $\triangle DEX$ and $\triangle BCX$ are isosceles right triangles, so $DX = 1$. Let $BX = x$. Then $BC = AB = x\sqrt{2}$.

The Pythagorean Theorem on right triangle $\triangle BDX$ gives $BD = \sqrt{x^2+1}$. Furthermore,

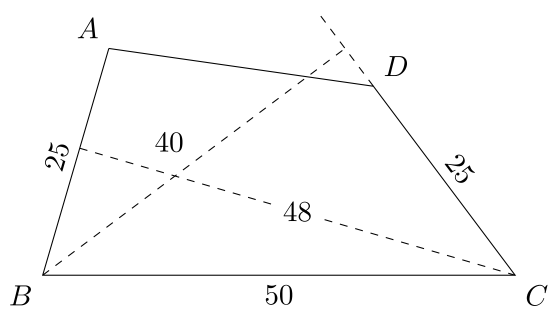
$$AB = AD + DB \implies x\sqrt{2} = \sqrt{2} + \sqrt{x^2+1}$$

which has candidate solutions $x = 2 \pm \sqrt{3}$. If $x = 2 - \sqrt{3}$, then $BD = (1-\sqrt{3})\sqrt{2} < 0$, which is impossible, so $x = 2 + \sqrt{3}$.

Finally,

$$\frac{AD}{AB} = \frac{\sqrt{2}}{\sqrt{2}(2+\sqrt{3})} = \boxed{\frac{1}{2+\sqrt{3}}} = \boxed{2-\sqrt{3}}.$$

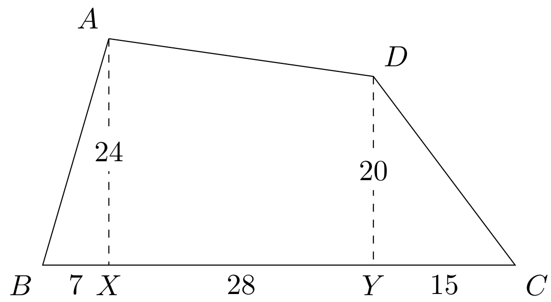
5. Quadrilateral $ABCD$ has $AB = CD = 25$, $BC = 50$, and acute angles at vertices B and C. The distance from C to line $\overleftrightarrow{AB}$ is 48, and the distance from B to line $\overleftrightarrow{CD}$ is 40. Compute the area of quadrilateral $ABCD$.



Answer: 850

Solution: First, we use the areas of triangles $\triangle ABC$ and $\triangle BCD$ to find the heights relative to the base $\overline{BC}$. Since $BC = 2AB$, the height from A to $\overline{BC}$ is half of the height from C to $\overleftrightarrow{AB}$, which is 24. Similarly, since $BC = 2CD$, the height from D to $\overline{BC}$ is half of the height from B to $\overleftrightarrow{CD}$, which is 20.

Let X and Y be the feet of the altitudes from A and D, respectively, to $\overline{BC}$. Then $AX = 24$ and $DY = 20$. Because the angles at B and C are acute, X and Y both lie on segment $\overline{BC}$. By the Pythagorean theorem, $BX = \sqrt{25^2 - 24^2} = 7$ and $CY = \sqrt{25^2 - 20^2} = 15$, so $XY = BC - BX - CY = 50 - 7 - 15 = 28$.



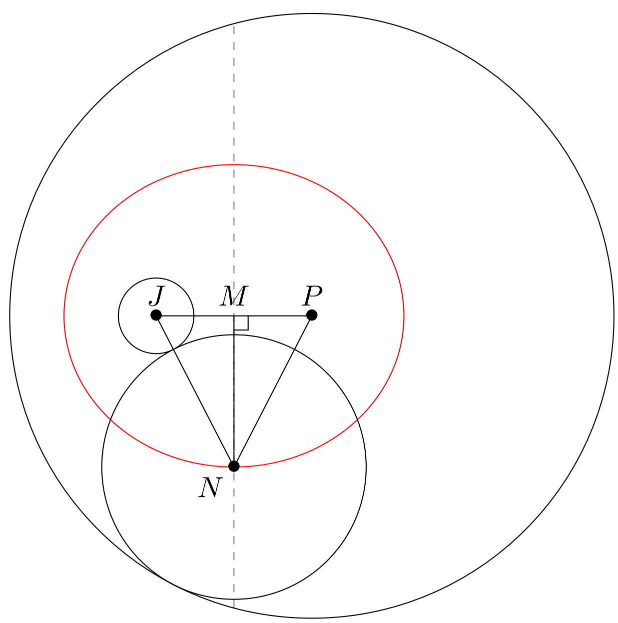
Then quadrilateral $ABCD$ can be partitioned into triangle $\triangle AXB$, triangle $\triangle DYC$, and trapezoid $AXYD$. The area of $\triangle AXB$ is $\frac{1}{2} \cdot 7 \cdot 24 = 84$, the area of $\triangle DYC$ is $\frac{1}{2} \cdot 15 \cdot 20 = 150$, and the area of trapezoid $AXYD$ is $\frac{24+20}{2} \cdot 28 = 616$. Thus, the area of quadrilateral $ABCD$ is $84 + 150 + 616 = \boxed{850}$.

6. Circles I and O have radii 1 and 8, respectively, and the distance between their centers is 4. Let $\mathcal{C}$ be the set of all circles simultaneously internally tangent to O and externally tangent to I. Compute the area enclosed by the curve formed from the centers of all circles in $\mathcal{C}$.

   Answer: $\boldsymbol{\frac{9\sqrt{65}}{4}\pi}$

   Solution: Let I have center J, and let O have center P. Let C be a circle in $\mathcal{C}$ with center X and radius r. Since C is externally tangent to I and internally tangent to O, we must have $JX = 1 + r$ and $PX = 8 - r$. Observe the sum $JX + PX = (1 + r) + (8 - r) = 9$ for any choice of C. Thus, the centers of the circles in $\mathcal{C}$ form the boundary of an ellipse with foci at J and P and major axis of length 9. Call this ellipse $L(\mathcal{C})$. The area of $L(\mathcal{C})$ is $\pi \cdot a \cdot b$ where a and b are the lengths of the semi-major and semi-minor axes, respectively. We know $a = \frac{9}{2}$, so it remains to find the length of the minor axis of the ellipse.

   Choose N on $L(\mathcal{C})$ such that N is on the perpendicular bisector of $\overline{JP}$, as shown below:



   Since N is on the perpendicular bisector of $\overline{JP}$, $NJ = NP = \frac{9}{2}$. Let M be the midpoint of JP.

By the Pythagorean theorem,

$$MN^2 = NJ^2 - MJ^2 = \left(\frac{9}{2}\right)^2 - \left(\frac{4}{2}\right)^2 = \frac{65}{4}.$$

The minor axis then has length $2MN = \sqrt{65}$, and the area of $L(\mathcal{C})$ is $\pi \cdot \frac{9}{2} \cdot \frac{1}{2}\sqrt{65} = \boxed{\frac{9\pi\sqrt{65}}{4}}$.

7. In three-dimensional coordinate space, a *lattice point* is a point whose coordinates are all integers. A triangular prism has all vertices at lattice points. Its triangular faces are both parallel to the xy-plane, and its rectangular faces are all parallel to the z-axis. The prism has 34 lattice points on its edges (including the vertices), 72 lattice points in the interior of its faces (not including the edges), and 126 lattice points in its interior (not including the faces). What is the volume of the prism?

Answer: 168

Solution: Let B and I be the numbers of lattice points on the boundary and interior of a triangular base, respectively, and let h be the height of the prism. Then the number of lattice points on the edges is $2B + 3(h-1) = 34$, the number of lattice points in the interior of the faces is $2I + (h-1)(B-3) = 72$, and the number of lattice points in the interior of the prism is $(h-1)I = 126$. Setting $H = h - 1$, we get the system

$$\begin{aligned} 2B + 3H &= 34, \\ 2I + H(B-3) &= 72, \\ HI &= 126. \end{aligned}$$

This system can be solved directly, but it is computationally intensive. To solve it more easily, we can leverage the fact that the variables are nonnegative integers. The first equation tells us that H is even and $H \leq 10$. Thus, the possibilities for H are $H = 2, 4, 6, 8, 10$, and the corresponding possibilities for B are $B = 14, 11, 8, 5, 2$. Plugging in these candidate values and solving for I shows that $(H, B, I) = (6, 8, 21)$ is the only consistent solution. By Pick's Theorem, the area of the triangular base is $I + \frac{B}{2} - 1 = 24$, so the volume of the prism is $24 \cdot h = 24 \cdot (H+1) = \boxed{168}$.

For completeness, one example of a triangle with vertices at lattice points that has 8 boundary points and 21 interior points is the triangle with vertices at $(0,0)$, $(6,0)$, and $(3,8)$.

8. In acute triangle $\triangle ABC$ with orthocenter H, let X be the unique point on $\overline{BC}$ for which angle $\angle AXH$ is maximized. Given $HX = 15, AX = 20$, and $BC = 30$, compute $AB + AC$.

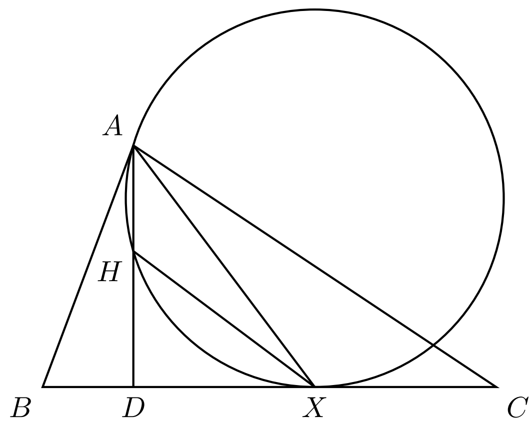
Answer: $\mathbf{8\sqrt{13} + 2\sqrt{73}}$

Solution:

We first claim that the circumcircle ω of $\triangle AXH$ is tangent to side BC.

Suppose that ω intersects side BC at points X and Y. Then, $\angle AXH = \angle AYH$. Furthermore, for any point Z in the interior of segment $\overline{XY}$, we have $\angle AYH < \angle AZH$ (consider inscribed angle theorem on the smaller circumcircle of $\triangle AZH$ where AH is fixed). Thus, moving points X and Y closer together increases $\angle AXH = \angle AYH$. The point X that maximizes the angle $\angle AXH$ is the point for which ω is tangent to BC.

Let D be the foot of the altitude from A to BC so that $AD \perp BC$, as shown below.



Since ω is tangent to side BC, we have $\angle DXH = \angle HAX = \angle DAX$. Both $\triangle DAX$ and $\triangle DHX$ are right triangles, so $\triangle DHX \sim \triangle DXA$. Hence,

$$\begin{aligned} &\frac{HX}{DH} = \frac{AX}{DX}, \\ \implies &\frac{15}{DH} = \frac{20}{DX}, \\ \implies &\frac{DH}{DX} = \frac{3}{4}. \end{aligned}$$

Since $\triangle DHX$ is right with hypotenuse 15, we have $DH = 9$ and $DX = 12$. Then, since $\triangle DAX$ is right with hypotenuse 20, $AD = 16$.

To help find AB and AC, we compute BD and CD. By a common fact*, $BD \cdot CD = DH \cdot DA = 9 \cdot 16 = 144$. Since $BD + CD = BC = 30, BD = 6$ and $CD = 24$. Finally, $AB + AC = \sqrt{16^2 + 6^2} + \sqrt{24^2 + 16^2} = \boxed{8\sqrt{13} + 2\sqrt{73}}$.

*To show this, construct ω_1, the circumcircle of $\triangle ABC$, and let H' be the reflection of H over side BC. Notice that $\triangle DHC \cong \triangle DH'C$, so $\angle H'CD = \angle DCH$. Then,

$$\angle H'CD = \angle H'CB = 90 - \angle B = \angle DAB = \angle H'AB,$$

so H' lies on ω_1. Power of a point at D with respect to ω_1 gives the desired result.

9. Triangle $\triangle ABC$ has $AB = 4$, $AC = 5$, and $BC = 6$. Points E and F are the midpoints of sides $\overline{AB}$ and $\overline{AC}$, respectively. Let P be the foot of the altitude from A to side $\overline{BC}$, and let H be the midpoint of $\overline{AP}$. The circumcircles of triangles $\triangle HEB$ and $\triangle HFC$ intersect at $K \neq H$. Compute AK^2.

Answer: $\mathbf{\frac{6400}{319}}$

Solution: Let O be the circumcenter of $\triangle ABC$, and let A' be the reflection of A across O. Draw the line ℓ through A' perpendicular to $\overline{BC}$, and let it meet the circumcircle ω again at Z. Finally, let K be the second intersection of line ZH with ω.

We first verify that K coincides with the point described in the problem. Observe that

$$\angle BKH = \angle BKZ = \angle BCZ = 180^\circ - \angle BAZ = 180^\circ - \angle BEH,$$

so quadrilateral $KBEH$ is cyclic. An analogous argument shows $KCFH$ is cyclic as well.

Next, extend $\overline{BH}$ and $\overline{CH}$ to meet ω again at X and Y, respectively. We claim A', E, Y are collinear (and similarly A', F, X are collinear). Indeed,

$$\angle YHE = \angle YCB = \angle YAB = \angle YAE,$$

so $YAHE$ is cyclic. Since $YAHE$ is cyclic and $\angle AHE = 90^\circ$, we have $\angle AYE = 90^\circ$. Also, AA' is a diameter of ω, so $\angle AYA' = 90^\circ$. Hence, Y, E, A' are collinear. An analogous argument shows X, F, A' are collinear.

Finally, we show E, F, K, A' are concyclic. Note

$$\angle EKF = \angle EKH + \angle FKH = \angle EBH + \angle FCH = \angle ABX + \angle ACY = \angle XA'Y = \angle FA'E,$$

so $EFKA'$ is cyclic.

A homothety centered at A with ratio 2 takes $\triangle AEF$ to $\triangle ABC$, so the circumcircle of $\triangle AEF$ is internally tangent to ω at A. Thus, the radical axes of the circumcircles of $\triangle ABC$ and $\triangle AEF$ and the circle passing through E, F, K, A' are the tangent to ω at A and the lines EF and $A'K$. By the Radical Axis Theorem, these three lines meet at one point; let this point of concurrency be T.

Set $x = AK^2$. By similarity of $\triangle A'KA$ and $\triangle AKT$,

$$x = AK^2 = TK \cdot A'K = \sqrt{TA^2 - x}\,\sqrt{AA'^2 - x} \quad \Longrightarrow \quad x^2 = (TA^2 - x)\,(AA'^2 - x).$$

Thus it suffices to compute

$$AA'^2 \quad \text{and} \quad TA^2.$$

Using Heron's formula, we can calculate that the area of $\triangle ABC$ is $\frac{15\sqrt{7}}{4}$, so the circumradius is $R = \frac{4\cdot5\cdot6}{15\sqrt{7}} = \frac{8}{\sqrt{7}}$, hence

$$AA'^2 = 4R^2 = \frac{256}{7}.$$

To find TA, we can use right triangle $\triangle AHT$ with $\angle AHT = 90^\circ$. We have

$$AH = \frac{1}{2} \cdot AP = \frac{1}{2} \cdot \frac{2 \cdot 15\sqrt{7}/4}{6} = \frac{5\sqrt{7}}{8}.$$

Then

$$\angle HAT = \angle HAB + \angle BAT = \angle ACB + (90^\circ - \angle ABC) = 90^\circ + C - B.$$

Applying the Law of Cosines gives $\cos(B) = \frac{9}{16}$, $\sin(B) = \frac{5\sqrt{7}}{16}$, $\cos(C) = \frac{3}{4}$, and $\sin(C) = \frac{\sqrt{7}}{4}$, so

$$\begin{aligned}
\cos(\angle HAT) &= \cos(90^\circ + C - B) \\
&= \sin(B - C) \\
&= \sin(B)\cos(C) - \cos(B)\sin(C) \\
&= \frac{5\sqrt{7}}{16} \cdot \frac{3}{4} - \frac{9}{16} \cdot \frac{\sqrt{7}}{4} \\
&= \frac{3\sqrt{7}}{32}.
\end{aligned}$$

Therefore,

$$TA^2 = \left(\frac{5\sqrt{7}/8}{3\sqrt{7}/32}\right)^2 = \left(\frac{20}{3}\right)^2 = \frac{400}{9}.$$

Substituting gives

$$AK^2 = x = \frac{TA^2 \cdot AA'^2}{TA^2 + AA'^2} = \frac{\frac{400}{9} \cdot \frac{256}{7}}{\frac{400}{9} + \frac{256}{7}} = \boxed{\frac{6400}{319}}.$$

10. Triangle $\triangle ABC$ has $AB = 3 + \sqrt{2}$ and $AC = 1 + \sqrt{2}$. Point D lies on side $\overline{BC}$ such that $AD = 2 - \sqrt{2}$. The internal angle bisector of $\angle BAD$ intersects the circumcircle of triangle $\triangle BAD$ at $X \neq A$, and the internal angle bisector of $\angle CAD$ intersects the circumcircle of triangle $\triangle CAD$ at $Y \neq A$. Given $\overline{XY} \parallel \overline{BC}$, compute XY.

Answer: $\boldsymbol{\frac{4}{\sqrt[4]{2}}}$ **or** $\boldsymbol{2\sqrt[4]{8}}$

Solution 1: Since the internal angle bisector of $\angle BAD$ bisects arc BD (not including A), X is the midpoint of arc BD. Likewise, Y is the midpoint of arc CD (not including A). Since the tangent at X is parallel to $\overline{BD}$, and $\overline{XY} \parallel \overline{BC}$, the tangent at X passes through Y. Likewise, the tangent at Y passes through X. Therefore, XY is the external common tangent of the two circumcircles.

Let $BD = x$ and $CD = y$. For generality, let $AB = c$, $AC = b$, and $AD = d$.

Let the centers of the circumcircles of triangles BAD and CAD be O_B and O_C respectively, and let their radii be r_B and r_C respectively. Draw $\overline{O_BX}$ and $\overline{O_CY}$, and let them intersect $\overline{BC}$ at M and N respectively. Then $\overline{O_BX}$ is the perpendicular bisector of $\overline{BD}$ and $\overline{O_CY}$ is the perpendicular bisector of $\overline{CD}$. So $XMNY$ is a rectangle, and thus

$$XY = MN = MD + ND = \frac{1}{2} \cdot BD + \frac{1}{2} \cdot CD = \frac{1}{2} \cdot BC.$$

So it suffices to find $BC = x + y$.

Applying Stewart's Theorem on triangle ABC and cevian $\overline{AD}$ gives

$$xy(x + y) + d^2(x + y) = b^2x + c^2y.$$

So we seek another equation relating x and y.

By the Extended Law of Sines, $2r_B = \dfrac{AB}{\sin \angle ADB}$ and $2r_C = \dfrac{AC}{\sin \angle ADC}$, and since $\sin \angle ADB = \sin(180^\circ - \angle ADC) = \sin \angle ADC$, we have

$$\frac{r_B}{r_C} = \frac{AB \sin \angle ADC}{AC \sin \angle ADB} = \frac{AB}{AC} = \frac{c}{b}.$$

Now, we can get our second equation using the fact that $MX = NY$. Using directed lengths along O_BX and O_CY,

$$MX = O_BX - O_BM = r_B - r_B \cos \angle DO_BM = r_B(1 - \cos \angle DAB)$$

and

$$NY = O_CY - O_CN = r_C - r_C \cos \angle DO_CN = r_C(1 - \cos \angle DAC).$$

By the Law of Cosines, $\cos\angle DAB = \dfrac{c^2+d^2-x^2}{2cd}$ and $\cos\angle DAC = \dfrac{b^2+d^2-y^2}{2bd}$. Therefore,

$$\begin{aligned}\frac{r_B}{r_C} &= \frac{1-\cos\angle DAC}{1-\cos\angle DAB}\\ &= \frac{1-\frac{b^2+d^2-y^2}{2bd}}{1-\frac{c^2+d^2-x^2}{2cd}}\\ &= \frac{2cd(2bd-b^2-d^2+y^2)}{2bd(2cd-c^2-d^2+x^2)}\\ &= \frac{c}{b}\cdot\frac{y^2-(b-d)^2}{x^2-(c-d)^2}.\end{aligned}$$

Since $\dfrac{r_B}{r_C} = \dfrac{c}{b}$, we simply have $\dfrac{y^2-(b-d)^2}{x^2-(c-d)^2} = 1$, or rearranging,

$$x^2-y^2 = (c-d)^2-(b-d)^2 = c^2-b^2-2cd+2bd.$$

Rearranging the equation from Stewart's Theorem, we have the system of equations:

$$\begin{aligned}xy(x+y) &= (b^2-d^2)x+(c^2-d^2)y\\ x^2-y^2 &= c^2-b^2-2cd+2bd\end{aligned}$$

Factoring $x^2-y^2=(x+y)(x-y)$, dividing the first equation by the second, and cross-multiplying, we get:

$$\begin{aligned}xy(c^2-b^2-2cd+2bd) &= ((b^2-d^2)x+(c^2-d^2)y)(x-y)\\ &= (b^2-d^2)x^2+(c^2-b^2)xy-(c^2-d^2)y^2.\end{aligned}$$

Rearranging gives

$$(b^2-d^2)x^2+(2bd-2cd)xy-(c^2-d^2)y^2 = 0.$$

Amazingly, this factors as:

$$((b-d)x+(c-d)y)((b+d)x-(c+d)y) = 0.$$

Therefore, either $(b-d)x+(c-d)y=0$ or $(b+d)x-(c+d)y=0$. Here, $b>d$ and $c>d$, so $(b-d)x+(c-d)y=0$ isn't possible. Thus, $(b+d)x=(c+d)y$.

For convenience, let $z = \dfrac{x}{c+d} = \dfrac{y}{b+d}$, so that $x=z(c+d)$ and $y=z(b+d)$. Then substituting into the second equation gives $z^2(c+d)^2-z^2(b+d)^2 = (c-d)^2-(b-d)^2$, or

$$z^2 = \frac{(c-d)^2-(b-d)^2}{(c+d)^2-(b+d)^2}.$$

We can now find the answer from this point: we can substitute $c = AB = 3+\sqrt{2}$, $b = AC = 1+\sqrt{2}$, and $d = AD = 2-\sqrt{2}$ to get

$$z^2 = \frac{(2\sqrt{2}+1)^2-(2\sqrt{2}-1)^2}{5^2-3^2} = \frac{(4\sqrt{2})(2)}{16} = \frac{1}{\sqrt{2}},$$

so $z = \frac{1}{\sqrt[4]{2}}$. Then

$$BC = x + y = z(b + c + 2d) = 8z = \frac{8}{\sqrt[4]{2}},$$

so $XY = \frac{1}{2} \cdot BC = \boxed{\frac{4}{\sqrt[4]{2}}}$.

Solution 2: There is a more elegant way we could get the $x(b+d) = y(c+d)$ using inversion, bypassing the need for the use of trigonometry and much of the algebra in the first solution. (The rest of the solution remains the same.) Here's an outline:

- The parallel lines BC and XY suggest using inversion. Invert about point A with arbitrary radius r_1. In the resulting diagram with primes added to each point (note that $A' = A$), lines BC and XY create the circles $(A'B'D'C')$ and $(A'X'Y')$, where $(A'X'Y')$ is the D'-mixtilinear incircle of triangle $B'C'D'$.
- Now invert the diagram again, this time about D' with arbitrary radius r_2. In the resulting diagram with primes added to each point, the circle $(A'B'D'C')$ becomes a line $B''A''C''$, and the circle $(A'X'Y')$ becomes circle $(A''X''Y'')$ tangent to line $B''A''C''$. Common tangents gives

$$D''B'' + B''A'' = D''B'' + B''X'' = D''X'' = D''Y'' = D''C'' + C''Y'' = D''C'' + C''A''.$$

- Converting this fact $D''B'' + B''A'' = D''C'' + C''A''$ backwards through the inversion about D' gives

$$B'D' \cdot (D'A' + A'C') = C'D' \cdot (D'A' + A'B').$$

  Converting this fact backwards through the inversion about A gives

$$BD \cdot (AD + AC) = CD \cdot (AD + AB).$$

  This gives $x(b+d) = y(c+d)$, as desired.